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Price Dynamics in Limit Order Markets

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This article presents a one-tick dynamic model of a limit order market. Agents choose to submit a limit order or a market order depending on the state of the limit order book. Each trader knows that her order will affect the order placement strategies of those who follow and the execution probability of her limit order is endogenous. All traders take this into account which, in equilibrium, generates systematic patterns in transaction prices and order placement strategies even with no asymmetric information.

One of the most common ways in which traders exchange securities is in markets based on a limit order book. In a limit order market investors can post price-contingent orders to buy/sell at preset limit prices. Exchanges which operate in this fashion are the Paris Bourse, Tokyo, Toronto, and Sydney.[1] Despite the prevalence of the limit order system, the dynamic aspects of the limit order book have not been well explored.

This article presents an explicitly dynamic model of the limit order book in a one-tick market. Traders with different valuations for an asset arrive randomly at a marketplace and trade either immediately by submitting a market

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[1] The specialist on the New York Stock Exchange (NYSE) maintains a limit order book. This market is a complex hybrid with the estimates of the specialist as a counterparty on average in Hasbrouck and Sofianos (1993) of 13%. Madhavan and Sofianos (1997) demonstrate a wide variety in specialist participation. If applied to the NYSE, this model is most appropriate for large stocks where specialist participation is low.

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order or choose a better price at the risk of nonexecution by submitting a limit order. A limit order is only executed when enough market orders arrive during the remainder of the day to execute all preceding orders in the book that have time priority. So the endogenous probability of execution depends both on the state of the book when the trader submits her order, and how many market orders she believes will arrive over the remainder of the day. Thus when a trader makes her decision she explicitly takes into account how her order affects the incentives of future traders to submit either market or limit orders.

The fact that both the past through the state of the book and the future through expected order flow affect the placement strategy of individual agents means that there are systematic patterns in any generated transaction data. Hence, even in the absence of asymmetric information and with a random arrival of trader types over the course of the day one will observe nonrandom patterns in the transactions data and order placement strategies.

Specifically, if the possible orders that can be observed are a market sell, a limit sell, a market buy, and a limit buy, then

(i) The probability that the next transaction is at the ask is larger following a transaction at the ask than a transaction at the bid.

(ii) The probability of observing a limit buy order given that the last order was a limit buy order is smaller than the probability of observing a limit buy order after any other order or transaction.

(iii) The probability of observing a limit sell order after a transaction at the ask is greater than the probability of observing a limit sell order after any other transaction or order.

(iv) The probability of observing a limit sell order after a limit sell order is smaller than the probability of observing a limit sell order after a transaction at the bid which is smaller than the probability of observing a limit sell order after a limit buy order, which itself is smaller than the probability of observing a limit sell order after a transaction at the ask.

By characterizing the decision problem faced by an agent in a dynamic limit book market, the article explicitly shows how both sides of the market affect a trader's decision problem. That is, a trader who arrives at the market looks at *both* sides of the book to determine her optimal order strategy.

Financial economists now have access to order-level data. In order to understand these data we need models of price and transaction dynamics. To identify abnormal patterns we need an understanding of normal patterns. By explicitly linking the state of the limit order book to order submission strategies this article identifies the normal full–information characteristics of a continuous double auction with price contingent orders.

Recently Glosten (1994), Kumar and Seppi (1994), Chakravarty and Holden (1995), Rock (1996), and Seppi (1997) have developed static equilibrium models of the limit order book.[2] In particular, Seppi (1997) shows how competition between liquidity providers with different costs such as the monopolist specialist on the NYSE, limit order submitters, and a trading crowd result in a limit order book and how this affects market order traders. His focus is neither the choice between limit orders and market orders nor the dynamic evolution of the book.

Static games that analyze the limit order book, by their very structure, cannot provide us with any insight about how the flow of orders into and out of the limit book is affected by the current book.[3] Also, a static game does not allow an explicit role for time priority. The existence of time priority (as a rationing rule) is one of the reasons why limit orders are submitted instead of monitoring the market for a good time to trade. Thus it is important to determine how the state of the limit book affects an agent's order submission strategy.

One other article has dealt with a dynamic equilibrium model of the limit order book: Foucault (1993). The structure of his model is such that the limit order book is either empty or full. So the effect of marginal changes on the limit order book cannot be examined: if a trader at time t submitted a limit sell order then a trader at time $t+1$ cannot. One could interpret this as an assumption that all traders are massive relative to the book.

Empirical work (until recently) has been hampered by the lack of order-level data. Biais, Hillion, and Spatt (1995), Hamao and Hasbrouck (1995), and Harris and Hasbrouck (1996) describe empirical properties of limit orders in Paris, Tokyo, and New York, respectively. Hollifield, Miller, and Sandas (1996) and Sandas (1996) carry out a structural estimation on Swedish data. None of this work can be interpreted as a direct test of my model, but some of the observed patterns are consistent. For example, Biais, Hillion, and Spatt find that successive transaction prices are positively correlated.

In Section 1, the model is presented. In Section 2, each agent's optimal strategy is characterized and the equilibrium is presented. In Section 3, the implications of the equilibrium are presented. A simple example follows. Section 4 discusses the extension of the model to multiple units and multiple prices. Section 5 concludes.

1. The Model

There are two goods, consumption on day 1, C_1, and consumption on day 2, C_2.[4] There is an asset that on day 2 pays out V units of C_2 per share. Day 1 is

[2] Partial equilibrium models have been analyzed by Angel (1992) and Harris (1994).

[3] Research in the auction literature suggests that results of batch auctions do not translate into continuous auctions.

[4] The structure of the economy is similar to Glosten and Milgrom (1985), with a limit book but absent a stochastic asset.

market day and is broken down into $T+1$ distinct periods: $t = 0, \ldots, T$. In the marketplace, claims to the asset can be exchanged for C_1.[5] Traders can buy or sell claims to the asset. They are risk neutral and have preferences over C_1 and C_2 given by

$$U(C_1, C_2; \beta) = C_1 + \beta C_2.$$

At each period during day 1, one agent arrives at the market. This captures the fact that except at open and close, trades on exchanges are processed individually. Agents arrive sequentially and each arrives only once. The agent who arrives at period t is characterized by two attributes. The first is β_t, her personal trade-off of day 1 and day 2 consumption. β is drawn from a continuous distribution $F(\cdot)$ and can take on values in $[\underline{\beta}, \bar{\beta}]$ where $0 \leq \underline{\beta} \leq 1 \leq \bar{\beta}$. Second, with probability π_S the trader is a seller and has one unit of the good which she can sell, denoted -1. With probability π_B the trader is a buyer and has a unit of the asset she can buy, 1. With probability $1 - \pi_S - \pi_B$ the trader is neither. The results easily extend to the case where $F(\cdot)$ and the probabilities that traders have buy or sell endowments are not independent and not time invariant; however, they are always common knowledge.

The parameter β is a trader's personal trade-off between C_1 and C_2. It determines an agent's willingness to trade. To see this, suppose a trader buys one share of the asset. If she does so, she values it as βV. She will therefore purchase the asset if the price (denominated in units of C_1) is less than the benefit, that is, if $\beta V - \text{price} > 0$.

The β parameter can be interpreted as a degree of patience for trade. Hence traders with extreme β valuations want to trade immediately — they demand liquidity.[6] Traders with intermediate values of β are less desperate to trade and supply liquidity. It is important to recognize that [as in Glosten and Milgrom (1985)[7]] this renders the security market a private values auction. The implications of this assumption will become evident both in the discussion of each trader's optimal order submission strategy and in the position of the latent value of the asset in the two-tick market. In order to focus on the importance of time priority in determining a trader's optimal order submission choice, I restrict attention to a market in which there are only two prices, a bid B and an ask A.

A trader who arrives at the market at time t has only one opportunity to submit orders. Once she submits an order, it cannot be withdrawn.[8] The

[5] Prices in this market are exchange ratios (i.e., $\frac{C_1}{C_2}$).

[6] Alternatively, β can be interpreted as a subjective valuation of the asset: agents have different priors over the value of the asset tomorrow, V.

[7] They have a private and common value component.

[8] Markets do allow cancellation. For tractability I abstract from this and multiple entry.

market she enters at time t is characterized by the designated bid and ask price $\{B, A\}$ and a limit order book b_t.

In a dynamic market, if prices are continuous then traders can undercut by arbitrarily small amounts and obtain priority over traders who arrived earlier in the day. To motivate real time priority, in this model, as in all real exchanges, prices are discrete. The price tick is 1.

An arbitrarily large quantity of the asset can be sold at the bid and an arbitrarily large quantity of the asset can be bought at the ask. One can interpret this as either the existence of a trading crowd that is willing to provide a substantial amount of liquidity at the quotes [as in Seppi (1997)] or as the existence of a market maker who is responsible for providing depth at the quotes.[9] Alternatively, this can be viewed as a competitive market after price discovery. Any limit orders posted at the bid and ask prices have priority over the institutional investors.

The existence of institutional traders providing liquidity at the bid and the ask precludes price competition among limit orders. Any buy orders below the bid or sell orders above the ask have a zero probability of execution as all incoming orders are executed at a better price. Such orders will never be posted in equilibrium.[10] This simplification isolates the effect of time priority on limit order submission.

If a trader submits a market sell order, it transacts at the bid, and if a trader submits a market buy order, it transacts at the ask. If the trader submits a limit sell order at the ask, then the order will be executed if in the periods remaining, that is from $t + 1, \dots T$, the number of buy orders that arrive is larger than the number of limit sell orders at the ask that have time priority.

Given this structure, a trader who arrives at the market is left with a choice between a limit order at a prespecified price, a market order with a price one tick worse, or remaining out of the market. In other words, the trader who arrives at time t can opt for immediate execution (submit a market order), can opt for delayed execution (submit a limit order),[11] or can choose not to trade. The choice that she makes will critically depend on the limit order book that she faces.

Unexecuted limit orders are known collectively as the "book." The book at t, b_t, is a pair, giving at the bid B and the ask A the totals of unfilled orders that were submitted prior to t. A typical element of b_t is (b_t^B, b_t^A). b_t^B is the accumulated limit buy orders, so $b_t^B > 0$. b_t^A represents accumulated sell orders, so $b_t^A < 0$. In either case, the book associates with each price

[9] The Registered Trader in Toronto provides such price continuity.

[10] This assumption accords well with the observed transaction data on, for example, the NYSE, where 98.2% of transactions occurred with no change or a 1/8 point variation [*NYSE Fact Book* (1996)]. Further, Harris and Hasbrouck (1996) observe that in their superDOT sample, the modal limit order price is at the best quote.

[11] I do not distinguish between market orders and marketable limit orders.

a quantity of shares from past limit orders submitted at that price that have not been executed.

To illustrate the mechanics of the book, consider how it evolves for an arbitrary (not necessarily equilibrium) strategy. Suppose the book at the end of time t is given by (b_t^B, b_t^A). Depending on the action taken by the trader at time $t+1$, the following books are possible at the end of time $t+1$:

$$(b_{t+1}^B, b_{t+1}^A) = \begin{cases} (b_t^B - 1, b_t^A) & \text{trader } t+1 \text{ submits a market sell order} \\ (b_t^B, b_t^A - 1) & \text{trader } t+1 \text{ submits a limit sell order} \\ (b_t^B, b_t^A) & \text{no trade occurs} \\ (b_t^B + 1, b_t^A) & \text{trader } t+1 \text{ submits a limit buy order} \\ (b_t^B, b_t^A + 1) & \text{trader } t+1 \text{ submits a market buy order} \end{cases}$$

For a given book at time t, the action of the trader at time $t+1$ can either lengthen the book at one of the quotes or shorten it. If the trader shortens the book, she also changes the priority of existing orders on the book. The evolution of the book is illustrated in Figure 1.

The gains from trade (payoff) for any agent arriving at $t < T$ is determined by the future course of play, which specifies for each time period $t+1, \ldots, T$, the action of the agent indexed by that period. The future course of play is a random variable because it depends on the random arrival of subsequent agent types and the strategies that they follow. The fact that the payoff to each trader depends on the actions of subsequent traders in the market is important because it ensures that traders try to rationally anticipate the effect of their orders on the subsequent course of play. That is, in equilibrium the trader at time t submits orders optimally given her type and the book at time t and has correct beliefs about the distribution over the future values of the exogenous variables and the strategies of other traders. A strategy for an agent is denoted $\phi_t^b(b_t, \beta_t)$ for a buyer and $\phi_t^s(b_t, \beta_t)$ for a seller.

It is important to notice that the strategy of each trader depends on time, that is, the strategies are nonstationary. A finite horizon model best captures intraday trading because settlement today is different from settlement tomorrow. Intuitively, the behavior of a trader with 50 seconds to go until the closing bell differs from the strategy of a trader with 5 hours to go.[12]

This market can be formalized as a stochastic sequential game. The game is played by nature and a set of players, each indexed by the period in which she may arrive. When an agent arrives at the market, she observes the current state of the market, summarized by the book. In general one would not expect the same equilibrium outcome in markets in which traders cannot condition on the book. This assumption was chosen because the limit book

[12] Harris and Hasbrouck (1996) observe that in their sample of superDOT orders, 82% of the limit orders are day orders, that is, they expire at the close of the day's trading.

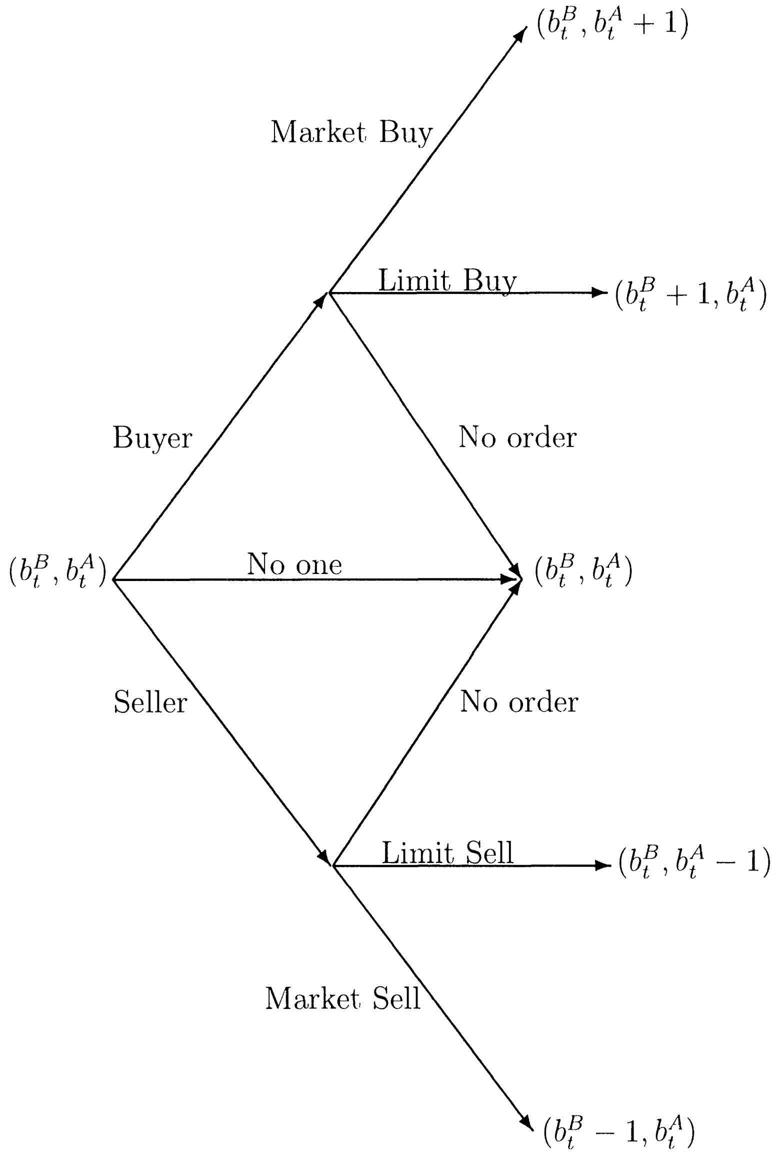

Figure 1
The evolution of the limit order book

Table 1
Notation

V	Value of the asset, scalar
β	Agent's subjective valuation of the asset $\beta \sim [\underline{\beta}, \bar{\beta}]$, continuous
B	The prevailing bid price
A	The prevailing ask price
b_t	The book at time t, a pair
(b_t^B, b_t^A)	Book at the bid and ask, $b_t^B > 0, b_t^A < 0$
-1^B	A market sell at the bid
-1^A	A limit sell at the ask
1^B	A limit buy at the bid
1^A	A market buy at the ask
p_t^s	The probability that a limit sell order executes by T
$p_t^s(b_t)$	The equilibrium probability that a limit sell order executes by T
p_t^B	The probability that a limit buy order executes by T
$p_t^B(b_t)$	The equilibrium probability that a limit buy order executes by T
$\bar{\beta}_t^b$	The highest β type who limit buys
$\underline{\beta}_t^s$	The lowest β type who limit sells
$\bar{\beta}_t^S$	The highest β type who limit sells
$\underline{\beta}_t^B$	The lowest β type who limit buys
$\bar{P}_t$	The observed transaction at time t

in Paris, Toronto, and Tokyo is open. So, as in those markets, this is a model of a transparent asset market.[13]

In this game, for each parameter formulation an equilibrium always exists because of the recursive structure. Given any state, the trader at time T faces a finite number of choices so that an optimum exists. The trader at time $T-1$, given any state and taking into account the effect of his actions on the trader at T, has an optimal order submission strategy and so forth. As a trader is indifferent between choices with zero probability, the equilibrium is unique. See Table 1 for notations.

2. Analysis of Equilibrium

Consider the queuing problem faced by a potential limit order submitter. When a market order arrives, limit orders are executed in the order in which they were submitted. Suppose a trader knows the probability that a fixed number of market orders will arrive over the rest of the day. If there are a large number of limit orders already on the book, then the probability of execution of her limit order is low. If this backlog of orders with time priority is sufficiently large then the investor might prefer to submit a market order.

First, to characterize an agent's optimal order submission strategy for a given limit order book, the probability that an additional limit order will execute by the end of the day is taken as given. Second, to characterize

[13] The degree of transparency is an important design feature and has been examined by Biais (1993), Madhavan (1996), and Pagano and Roell (1996), among others.

equilibrium, restrictions on the endogenous probability distribution are determined.

2.1 Agent's choice problem

Lemma 1 characterizes the solution to an agent's choice problem. There are two possible prices at which traders submit orders, B and A. If a trader is a seller, then she submits either a market order at the bid, a limit order at the ask, or she remains out of the market. If she submits a market order at the bid, then her order is (-1^B). If she submits a limit order, her order is (-1^A). That is, orders are designated by signed order flow and the price.

The agent's choice problem is characterized by associating to each trader an optimal action that depends on the realization of her β type — which is her willingness to trade — and if she is a buyer or a seller.

Consider an arbitrary limit sell order submitted at time $t < T$ at the ask, A, when the book is b_t. If the order is executed before the end of the day, then the net utility gain (payoff) to the order is $(A - \beta_t V)$. Ex ante, the trader anticipates a probability of execution for her order. This depends on the current book and the time period.

Denote the probability that a sell order entered into a book b_t at time t gets executed by time T as p_t^s. Similarly, denote the probability that a buy order entered in a book b_t at time t gets executed by T as p_t^b. It is important to recognize that these probabilities are assigned by the traders. In *equilibrium*, these probabilities will be correct given the exogenous distributions and the equilibrium trading strategies of other traders.

To determine if she wants to submit a market order or a limit order, a trader compares the expected utility gain she gets if she submits a limit sell order, $p_t^s[A - \beta V]$, with the utility gain that she receives if she submits a market order, $[B - \beta V]$. She chooses the one that gives her the highest reward. If she is indifferent between the two actions she has a cutoff β.

Lemma 1. *At period t, if the trader is a seller, then there exist cutoffs* $\underline{\beta}^s(p_t^s) \leq \frac{B}{V} \leq \bar{\beta}^s(p_t^s)$ *such that if*

$$\beta_t \in \begin{cases} [\underline{\beta}, \underline{\beta}_t^s(p_t^s)), & \phi_t^s(b_t, \beta_t) = -1^B, \quad \textit{(market sell)} \\ [\underline{\beta}_t^s(p_t^s), \bar{\beta}_t^s(p_t^s)) & \phi_t^s(b_t, \beta_t) = -1^A, \quad \textit{(limit sell)} \\ [\bar{\beta}_t^s(p_t^s), \bar{\beta}] & \phi_t^s(b_t, \beta_t) = 0, \quad \textit{(out of market)} \end{cases}$$

If the trader is a buyer, then there exist cutoffs $\underline{\beta}_t^b(p_t^b) \leq \frac{A}{V} \leq \bar{\beta}_t^b(p_t^b)$ *such that if*

$$\beta_t \in \begin{cases} [\underline{\beta}, \underline{\beta}_t^b] & \phi_t^b(b_t, \beta_t) = 0, \quad \textit{(out of market)} \\ (\underline{\beta}_t^b(p_t^b), \bar{\beta}_t^b(p_t^b)] & \phi_t^b(b_t, \beta_t) = 1^B, \quad \textit{(limit buy)} \\ (\bar{\beta}_t^b(p_t^b), \bar{\beta}] & \phi_t^b(b_t, \beta_t) = 1^A, \quad \textit{(market buy)} \end{cases}$$

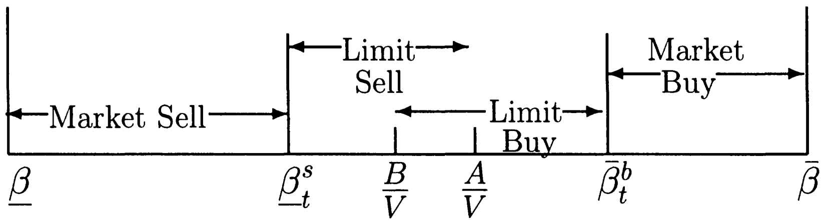

Figure 2
An agent's order choice

The actual values of $\underline{\beta}_t^s(p_t^b)$, $\bar{\beta}_t^s(p_t^s)$, $\underline{\beta}_t^b(p_t^b)$, $\bar{\beta}_t^b(p_t^b)$ at any time t will depend on the parameters of the problem and on the probability of execution. The cutoff βs between submitting market orders and limit orders are defined in terms of these execution probabilities. Specifically,

$$\underline{\beta}_t^s(p_t^s) = \max\left[\frac{A}{V} - \frac{\frac{1}{V}}{[1 - p_t^s]}, \ \underline{\beta}\right]$$

and

$$\bar{\beta}_t^b(p_t^b) = \min\left[\frac{B}{V} + \frac{\frac{1}{V}}{[1 - p_t^b]}, \ \bar{\beta}\right].$$

The cutoff βs between submitting limit orders and remaining out of the market are determined by how large the trader's valuation of the asset is relative to its price. The highest β type who would limit sell is the one who values the asset at $\frac{A}{V}$ and the lowest β type who would limit buy is the one who values the asset at $\frac{B}{V}$, therefore

$$\bar{\beta}_t^s(p_t^s) = \begin{cases} \frac{A}{V} & \text{if } p_t^s > 0 \\ \frac{B}{V} & \text{if } p_t^s = 0 \end{cases}$$

and

$$\underline{\beta}_t^b(p_t^b) = \begin{cases} \frac{B}{V} & \text{if } p_t^b > 0 \\ \frac{A}{V} & \text{if } p_t^s = 0. \end{cases}$$

Equilibrium trade choices are illustrated in Figure 2.

Notice that the more extreme β types submit market orders. If β is very low then the trader does not value consumption tomorrow very highly, and when she sells she is willing to accept a lower price if she is guaranteed execution. Such β types demand liquidity and will pay a tick to get it. By contrast, traders who submit limit orders have β realizations close to 1, that is, they are nearly indifferent between consumption today and consumption tomorrow. They therefore require a higher price to induce trade and so submit limit orders. Such β types supply liquidity. Of course, the definitions of high and low depend on the state of the book.

Each trader can only trade on one side of the market. However, notice that there is a range of βs, specifically $\left(\frac{B}{V}, \frac{A}{V}\right)$, who would place limit orders to both buy *and* sell *if* they had the endowment. This is because a trader values the asset at βV. If she limit sells, she does so at price A. If she limit buys, she does so at price B. Therefore if she submits limit orders, a trader can simultaneously buy low and sell high if her personal valuation of the asset is contained within the spread.

From the definitions of the marginal βs, anything that increases the probability of limit order execution increases the range of β types who submit limit orders and decreases the range of β types who submit market orders.

Lemma 2. *The higher the probability of execution of a limit order, the more trader types prefer to submit limit orders over market orders. That is,*

$$\frac{d\underline{\beta}_t^s(p_t^s)}{dp_t^s} \leq 0$$
$$\frac{d\bar{\beta}_t^b(p_t^b)}{dp_t^b} \geq 0.$$

Any trader explicitly takes into account the effect that her trade will have on subsequent orders: by changing the queue, a trader at time t can change the decision problem faced by traders who arrive later in the day. This effect is the basis for strategic behavior. To understand how transactions at time t affect transactions in subsequent periods one needs to determine how the transaction probability changes. Specifically, how changes in the thickness of the book on both sides of the market affect the transaction probability. These effects are captured in Proposition 1.

Proposition 1 reflects time priority: limit orders are tagged according to their place in the queue. Hence the length of the queue matters. Proposition 1 also formalizes the notion that *both* sides of the book affect trade choices. This is because the book on the other side of the market can affect a potential counterparty's decision to submit a market order. For example, if a seller at time t sees that there are not many buy orders on the book at the bid, then she knows that in subsequent periods buyers are more likely to submit limit buy orders and less likely to submit market buy orders. This means that any limit sell order that she submits will have a lower chance of being executed. Hence, even in this simple environment, a trader must take both sides of the market into consideration if she places a limit order.

Proposition 1. *In equilibrium, at any time t, if the book at the ask is one unit thinner, then the probability of execution of a limit sell order is higher. If the book at the bid is one unit thicker, then the probability of execution of a limit sell order is greater. If the book is one unit thicker at the bid and one unit thicker at the ask, then the probability of execution of a limit sell order is lower. Symmetric conditions hold for the probability of execution of a limit buy order. Hence,* $\forall\, b_t, t,$

(i) $p_t^s(b_t) \leq p_t^s(b_t^B, b_t^A + 1)$
(ii) $p_t^s(b_t) \leq p_t^s(b_t^B + 1, b_t^A)$
(iii) $p_t^s(b_t) \geq p_t^s(b_t^B + 1, b_t^A - 1)$
(iv) $p_t^b(b_t) \leq p_t^b(b_t^B - 1, b_t^A)$
(v) $p_t^b(b_t) \leq p_t^b(b_t^B, b_t^A - 1)$
(vi) $p_t^b(b_t) \geq p_t^b(b_t^B + 1, b_t^A - 1)$

So, for example, (v) compares two possible books at time t, (b_t^B, b_t^A) and $(b_t^B, b_t^A - 1)$. The latter has one more limit order at the ask than the former. Proposition 1 states that $p_t^b(b_t) \leq p_t^b(b_t^B, b_t^A - 1)$, that is, in equilibrium, the probability is lower that a limit buy order submitted at time t is executed by the end of the day if it is submitted in the book (b_t^B, b_t^A) than in the book $(b_t^B, b_t^A - 1)$.

The rankings in Proposition 1 are statements about the equilibrium behavior of traders.

3. Predictions

How changes in the book affect an agent's decision to submit orders is important because the simple mechanism of "crowding out" generates systematic patterns in the transactions data. This may seem counterintuitive: after all, with random draws of types each period and a fixed spread, one would not expect to observe any persistent relationships between transactions in adjacent periods. In a market with a limit book, however, there is a relationship between past and current order flows. This relationship allows us to explain Biais, Hillion, and Spatt's (1995) finding that on the Paris Bourse, small market orders to buy are less frequent after small market orders to sell than after small market orders to buy.[14]

The reasoning is as follows: suppose a trader at time t submits a market buy order. This is observed as a transaction at the ask and the book at the ask is ceteris paribus reduced by one unit. A limit sell order at the ask therefore has a better chance of execution. The trader at time $t + 1$, if she is a seller, is more inclined to submit a limit order because the payoff to limit orders increases in the probability of execution; therefore she is less likely to submit a market sell order that would appear as a transaction at the bid.

[14] It is inappropriate to evaluate empirical work on "large" orders with this model.

Proposition 2. *The probability of a transaction at time $t+1$ at the bid is larger if the transaction at time t was at the bid than if it was at the ask. (A symmetric result holds for the other side of the market.)*

If P_t denotes the observed transaction price at time t then

$$\Pr[P_{t+1} = B \mid P_t = A, b_t] \leq \Pr[P_{t+1} = B \mid P_t = B, b_t].$$

Proposition 2 refers to successive transaction prices, not to successive price changes. It is well known that with a fixed spread, the covariance between successive price changes must be negative. It is interesting to compare Proposition 2 with the analysis of the bid–ask bounce presented in Roll (1984). He presumes that buy or sell orders are equally likely to occur in any period. The change in transaction price with a fixed bid and ask is either the spread, $s = A - B$ in his notation, $-s$, or 0. He calculates that the covariance between successive price changes is $\frac{-s^2}{4}$. Clearly, if one considers the effect of the limit book on the decision of traders to submit market orders, a transaction at the bid followed by a transaction at the ask does not necessarily have the same probability as a transaction at the bid followed by a transaction at the bid. Therefore, while the same outcomes are possible as in Roll, the probability of each of the successive price changes is not constant. Specifically, continuations are more likely than reversals. The probability of observing no price change is larger than the probability of observing a price change and so the effective spread measure suggested by Roll is a lower bound, that is, $-2\sqrt{\text{cov(price changes)}} \geq s$.[15]

The intuition that the order submitted by the trader at time t affects the order submitted by a trader at time $t+1$ also appears as a type of "crowding out" mechanism. The trader at time t, by lengthening the queue at one of the quotes, can make limit orders unattractive to a subsequent trader.

Proposition 3. *The probability of observing a limit buy order at time $t+1$ is smaller if the transaction at time t was a limit buy order than if it was not. (A symmetric result holds for the other side of the market.)*

$$\begin{aligned} &\Pr[\phi^b_{t+1}(b_{t+1}, \beta_{t+1}) = 1^B \mid \phi^b_t(b_t, \beta_t) = 1^B, b_t] \\ &\quad \leq \Pr[\phi^b_{t+1}(b_{t+1}, \beta_{t+1}) = 1^B \mid \phi_t(b_t, \beta_t) \neq 1^B, b_t] \end{aligned}$$

It is difficult to compare this prediction directly to the empirical work of either Biais, Hillion, and Spatt (1995) or Hamao and Hasbrouck (1995) as neither of these articles considers order flow conditional on constant quotes. In both of these articles the quotes move over time. Intuitively, a movement

[15] Huang and Stoll (1996) use the Roll spread (among other measures) to compare transaction costs on the NYSE and NASDAQ. This model provides a theoretical justification for Choi, Salandro, and Shastri's (1988) extension of Roll's measure to include positive serial correlation.

in the quotes has two effects on limit order submission. For a sell order, if the bid decreases, then the order submitter has at least one more price at which she can submit a limit order or be undercut. Also, the price at which she can submit a market order is worse.

Examination of the limit order book allows us to determine the relationship between observed transactions and limit order submission: order flow attracts order flow. If a trader removes some of the supply of liquidity at the quotes by submitting a market order, then it becomes more attractive for the next trader to submit a limit order at the quotes. Hence limit sell orders are correlated with market buy orders and limit buy orders are correlated with market sell orders.

Proposition 4. *The probability of observing a limit sell order at time $t+1$ is larger if the transaction at time t was at the ask than if the transaction at time t was at the bid. (A symmetric result holds for the other side of the market.)*

$$\Pr[\phi^s_{t+1}(b_{t+1}, \beta_{t+1}) = -1^A \mid P_t = A, b_t] \\ \geq \Pr[\phi^s_{t+1}(b_{t+1}, \beta_{t+1}) = -1^A \mid P_t = B, b_t].$$

Lee, Mucklow, and Ready (1993) examine the relationship between quotes, quoted depth, and volume on the NYSE. In as much as the limit book affects the quotes of the specialist [as in Seppi (1997)], then bigger quoted depth corresponds to a thicker limit book. A period of high trading volume — a long sequence of market orders — must necessarily be associated with a thinner book and hence less quoted depth. This is the association that they find.

A simple way to distinguish between a random realization of a string of market orders and a true increase in trading activity in a particular security is to look at order–level data. Specifically, did the number of limit orders also increase? As they note, however, they do not examine the effect of quoted spreads and depth on liquidity. The logic of this article suggests that a thicker limit book through the mechanism of crowding out is ceteris paribus associated with more market orders.

Although both sides of the market are important in determining what kind of order a trader will submit, direct competition has more effect on order submission strategies than potential strategic effects. Consider two books where one has a longer queue at the ask. Suppose a seller submits limit orders in both cases. The longer queue has two effects. First, if a market order arrives, the limit sell order has a lower time priority and so requires the arrival of more market buy orders for execution. Second, the fact that the queue at the ask is longer induces fewer buyers to submit market orders — the buyers rationally anticipate the crowding out of limit sell orders and so limit buy orders become more attractive. Both of these effects work to reduce the probability of execution of a limit sell order when the queue is longer at the ask. Further, the combination of these two effects is larger than just the anticipated change in market buy orders if the books differ only the by length of the queue at the bid.

Since changes in the queues on different sides of the market have asymmetric effects on execution probabilities, one can explicitly rank the probability of seeing any specific action, say limit selling, conditional on the previous action:

Proposition 5. *The probability of observing a limit sell order at time $t+1$ given that the transaction at time t was a limit sell order is at most as large as the probability conditional on the time t transaction being a market sell order. This probability is in turn at most as large as the probability of observing a limit sell order at time $t+1$ conditional on the transaction at time t being a limit buy order. The probability of observing a limit sell order at time $t+1$ is largest when the transaction at time t is a market buy order. In other words:*

$$\begin{aligned}
&\Pr[\phi^s_{t+1}(b_{t+1}, \beta_{t+1}) = -1^A \mid \phi^s_t(b_t, \beta_t) = -1^A, b_t] \\
&\quad \leq \Pr[\phi^s_{t+1}(b_{t+1}, \beta_{t+1}) = -1^A \mid P_t = B, b_t] \\
&\quad \leq \Pr[\phi^s_{t+1}(b_{t+1}, \beta_{t+1}) = -1^A \mid \phi^b_t(b_t, \beta_t) = 1^B, b_t] \\
&\quad \leq \Pr[\phi^s_{t+1}(b_{t+1}, \beta_{t+1}) = -1^A \mid P_t = A, b_t].
\end{aligned}$$

The stylized queuing model allows strong predictions on patterns in the transactions data and on order submission strategies. Specifically, even in the absence of movement in the latent value of the asset, the limit book provides a link between order flows in successive periods. Of course, there are many frictions that could conceivably cause patterns in the data. For example, a large institutional trader might split up his market buy orders over time to minimize price impact. This might appear as a sequence of market buy orders. This model, however, predicts that while continuations are more likely than reversals, this is because sellers substitute away from market sell orders to limit sell orders. So joint observations on both transactions and order submissions should allow the econometrician to distinguish between the normal effects of a limit order book and more complex frictions.

3.1 A simple three-period example

This three-period example of the queuing model characterizes the equilibrium strategies of traders at period T, $T-1$, and $T-2$. It shows that at time T and $T-1$, the probability of execution of a limit order depends on the thickness of the book on the same side of the market. It also demonstrates that by time $T-2$, the thickness of the book on both sides of the market affect order submission strategies.

Consider the following three-period example of the queuing model. The asset value V is constant at 5.5. The tick size is 1 and the bid (B) is at 5 and the ask (A) is at 6. Trader's valuations of the asset are drawn from a uniform distribution on [0,2]: $\beta \sim U[0,2]$. The probability of a trader having either a sell endowment or a buy endowment is 1/2, so with probability 1/2 the trader can buy a unit and with probability 1/2 the trader can sell a unit.

At time T, traders only submit market orders. No limit order submitted at time T could be executed. So the only traders who will trade are buyers for whom $\beta V - A \geq 0$ or $\beta \geq \frac{6}{5.5} = 1.09$, and sellers for whom $B - \beta V \geq 0$ or $\beta \leq \frac{5}{5.5} = 0.91$.

The equilibrium behavior of the traders at time T determines the probability distribution over outcomes assessed at time $T-1$. Therefore the $T-1$ probability of observing one market buy order in the last period is

$$\left[\frac{2-1.09}{2-0}\right]\frac{1}{2} = 0.23.$$

A similar calculation yields the $T-1$ probability of seeing one market sell order in the last period as 0.23.

At time $T-1$, at most one market order could arrive before the end of the day, and so a trader only submits a limit order if the book at time $T-1$ is empty. A seller at time $T-1$, when faced with an empty book, puts in a limit sell order if the expected payoff to the limit order is larger than the payoff to a market order. That is, if

$$[6 - \beta 5.5][0.23] \geq [5 - \beta 5.5]$$

or $\beta \geq 0.86$. The highest valuation seller who would consider submitting a limit order is one for whom $[6 - \beta V] = 0$ or $\beta = 1.09$. A symmetric argument applies to buyers.

In summary, if the book at the ask is empty at time $T-1$, then if the trader is a seller for

$$\beta \in \begin{cases} [0, 0.86] & \text{market sell} \\ (0.86, 1.09] & \text{limit sell} \\ (1.09, 2] & \text{no trade,} \end{cases}$$

whereas if there is at least one unit in the book at the ask at time $T-1$, then if a trader is a seller for

$$\beta \in \begin{cases} [0, 0.91] & \text{market sell} \\ (0.91, 2] & \text{no trade.} \end{cases}$$

For a buyer, if the book at the bid is empty at time $T-1$, then if the trader is a buyer for

$$\beta \in \begin{cases} [0, 0.91] & \text{no trade} \\ (0.91, 1.15] & \text{limit buy} \\ (1.15, 2] & \text{market buy,} \end{cases}$$

whereas if there is at least one unit in the book at the bid at time $T-1$, then if a trader is a buyer for

$$\beta \in \begin{cases} [0, 1.09] & \text{no trade} \\ (1.09, 2] & \text{market buy.} \end{cases}$$

The decision of traders at time $T-1$ and T were dictated by the book. The book is *always* too full for the trader at time T, and if there is anything in it at time $T-1$ then no limit order is submitted. This is the starkest example of crowding out. Notice however that the only relevant feature of the book is the thickness at the bid for a buyer and the thickness at the ask for a seller.

At time $T-2$, the situation is more complex. Consider the probability that a limit sell order submitted at time $T-2$ will be executed by the end of the trading day. If the book at the ask has two or more units in it, then the probability of execution is zero, so no limit sell orders will be submitted.

If the book at the ask has only one unit in it, then a limit order submitted at the ask will be executed if a market buy order arrives both at time $T-1$ and at time T because the submitted limit sell order will be the second in line at the ask. The probability that an order submitted at time T is a market buy order is 0.23.

But what of the trader at time $T-1$? If the book at the bid has one unit in it, then if a buyer arrives and has $\beta \geq 1.09$, she submits a market order. However, if the book at the bid is empty, then a potential buyer at time $T-1$ has to decide between submitting a limit and a market order. Specifically, for $\beta \in [1.09, 1.15]$, if there is no order on the book at time $T-1$, she submits a limit order, but if there *is* an order on the book at time $T-1$, then she submits a market order. So if there is less depth in the book at the bid at time $T-2$, then the probability of execution of a limit sell order submitted at time $T-2$ is lower. Hence the probability of execution of a limit sell order submitted at time $T-2$ depends both on the orders at the bid and the ask.

4. Extension to Multiple Units and Multiple Prices

The model can readily be extended to the case where each buyer and each seller has multiple units to trade. If traders have multiple units, one has to take into account optimal order splitting. While considerably more complex, the intuition will still hold if the bid and the ask are one tick apart and each trader is either a buyer or a seller, but not both.[16]

Characterizing the evolution of the book when there are many prices becomes highly complex because the state space expands. Specifically, the larger the number of prices over which traders compete, the more complex the book. A complete characterization of the equilibrium requires a ranking of all execution probabilities for all possible states. That such an equilibrium exists follows immediately from the finite strategy space. Apart from a complete characterization, some interesting features in a market with some price competition deserve comment.

Consider a market in which a trading crowd sustains a spread of two ticks; that is, there is one price between their bid and ask. If there is one price between the crowd's bid and ask, at that price there could be a limit sell order, a limit buy order, or no order. The actual bid and the ask in the market will depend on the best limit buy and sell orders.

In such a market, the latent value of the asset V need not be in the interval between the best bid and the best ask. Consider a buyer who arrives at the market when there is no existing order between the trading crowd's bid and ask. In preference to submitting a market order at the ask she might submit a limit order at the intermediate price. This could result in a spread that does not contain the latent value of the asset. This, of course, is a consequence of the private values assumption.

In such a market it is interesting to notice that if there is a drop in the bid, then there also will be a drop in the ask. This dynamic was observed by Biais, Hillion, and Spatt (1995). Specifically,

Proposition 6. *In a two-tick market, the probability of seeing a drop in the ask after there has been a drop in the bid is greater than seeing the ask remain the same or increase. (A symmetric result holds for the other side of the market.) Specifically,*

$$\Pr[A_{t+1} \leq A_t \mid B_t < B_{t-1}] \geq \Pr[A_{t+1} > A_t \mid B_t < B_{t-1}].$$

In a two-tick market, if the bid drops, then the bid and the ask are a full two ticks apart. As the trading crowd maintains its presence at the old ask, it cannot rise. A seller arriving in the market can either submit an order at the old ask, undercut the old ask by a tick, or submit a market order. The probability of observing this must be weakly positive and hence weakly larger than the probability of observing the ask rise or remain the same.

Of course, such effects can be generated by a model with price discovery. In particular, with a Glosten and Milgrom (1985) market maker one would observe such dynamics. Also, in a different framework in which competing

[16] If traders have both buy and sell endowments the strategic environment is more complex. In particular, traders could submit multiple order types to manipulate subsequent traders.

dealers set spreads and execute trades, such dynamics could arise from inventory effects. However, this simple two-tick environment demonstrates that with a pure limit order book systematic price movements that appear to be correlated with information need not be.

5. Conclusion

This is a dynamic model of the evolution of the limit book. The optimal choice between submitting a limit order and a market order is characterized. In equilibrium, the dynamics of flows into and out of the limit order book are characterized.

The central intuition of the article is that each trader knows that her order will affect the order placement strategies of those who follow. She takes these effects into account, which, in equilibrium, generates systematic patterns in prices and order placement strategies. As the optimal choice of either market or limit orders by traders generate systematic patterns in any observed transaction data, an econometrician seeking to draw inferences from transaction-level data should take the systematic patterns generated by liquidity traders into account. In particular, both sides of the book are important in determining an agent's order choice. Hence both sides of the market should be examined when drawing inferences about correlations.

The strategic behavior of liquidity traders in the marketplace is economically important: all traders face these effects. Even with asymmetric information, the trade-off between price improvement and execution probability is pertinent. One could imagine an informed trader submitting a limit order to get the best possible price while rationally anticipating the future order flow. In order to understand informed trading behavior we also need to understand uninformed behavior. To analyze behavior within limit order markets we have to understand order submission strategies. Further, if we wish to identify what is abnormal about a particular price series, we need a model or an understanding of what normal is. This is a stylized model, but it specifies relationships between observables and can act as a benchmark.

Appendix

Proof of Lemma 1. If a trader limit sells, she does so at the ask A. If a trader limit buys, she does so at the bid B. Denote by p_t^s the probability of execution of a sell order submitted at time t when the book is b_t and denote by p_t^b the probability of execution for a buy order at time t, with the same book.

The possible actions that a seller can take are to submit a market order, -1^B; submit a limit order, -1^A; or remain out of the market. Therefore a seller at time t chooses

$$\max\left[[B - \beta_t V],\, p_t^s[A - \beta_t V],\, 0\right].$$

To characterize the solution in terms of a trader's β realization define

$$\underline{\beta}_t^s(p_t^s) = \begin{cases} \underline{\beta}, & \text{if } p_t^s \geq \frac{B-\underline{\beta}V}{A-\underline{\beta}V} \\ \text{solves } [A - \underline{\beta}_t^s V]p_t^s = [B - \underline{\beta}_t^s V], & \text{otherwise.} \end{cases}$$

$\underline{\beta}_t^s(p_t^s)$ is a lower bound on limit selling because in the second case $(A - \underline{\beta}_t^s(b_t)V)p_t^s$ decreases in β at rate $p_t^s V$, whereas $(B - \underline{\beta}_t^s(p_t^s))$ decreases at rate V.

Define

$$\bar{\beta}_t^s(p_t^s) = \begin{cases} \frac{A}{V} & \text{if } p_t^s > 0 \\ \frac{B}{V} & \text{if } p_t^s = 0 \end{cases}$$

Notice that

$$\underline{\beta}_t^s(p_t^s) = \max\left[\frac{A}{V} - \frac{\frac{1}{V}}{[1 - p_t^s]}, \underline{\beta}\right].$$

Therefore $\underline{\beta}_t^s(p_t^s) \leq \bar{\beta}_t^s(p_t^s)$.

The possible actions that a buyer can take are to submit a market order, 1^A; submit a limit order, 1^B; and remain out of the market. The trader chooses the action that gives him

$$\max\left[[\beta_t V - A],\, [\beta_t V - B]p_t^b,\, 0\right].$$

Hence, for limit buy orders,

$$\bar{\beta}_t^b(p_t^b) = \begin{cases} \bar{\beta}, & \text{if } p_t^b \geq \frac{\bar{\beta}V-A}{\bar{\beta}V-B} \\ \text{solves } [\bar{\beta}_t^b(p_t^b)V - B]p_t^b = \bar{\beta}_t^b(p_t^b)V - A, & \text{otherwise.} \end{cases}$$

This is an upper bound because in the second case $(\bar{\beta}_t^b(p_t^b)V - B)p_t^b$ increases in β at rate $p_t^b V$, while $\bar{\beta}_t^b(p_t^b)V - A$ increases in β at rate V.

Define

$$\underline{\beta}_t^b(p_t^b) = \begin{cases} \frac{B}{V} & \text{if } p_t^b > 0 \\ \frac{A}{V} & \text{if } p_t^b = 0 \end{cases}.$$

Therefore $\bar{\beta}_t^b(p_t^b) \geq \underline{\beta}_t^b(p_t^b)$. ∎

Proof of Lemma 2. Immediate. ∎

Proof of Proposition 1. This characterizes equilibrium strategies; as such, I describe them directly in terms of the book.

First notice that, by the definitions of $\bar{\beta}_t^b(b_t)$ and $\underline{\beta}_t^s(b_t)$, the following six statements correspond to statements (i)–(vi) of Proposition 1:

(i) $\underline{\beta}_t^s(b_t^B, b_t^A) \geq \underline{\beta}_t^s(b_t^B, b_t^A + 1)$

(ii) $\underline{\beta}_t^s(b_t^B, b_t^A) \geq \underline{\beta}_t^s(b_t^B + 1, b_t^A)$

(iii) $\underline{\beta}_t^s(b_t^B, b_t^A) \leq \underline{\beta}_t^s(b_t^B + 1, b_t^A - 1)$

(iv) $\bar{\beta}_t^b(b_t^B, b_t^A) \leq \bar{\beta}_t^b(b_t^B - 1, b_t^A)$

(v) $\bar{\beta}_t^b(b_t^B, b_t^A) \leq \bar{\beta}_t^b(b_t^B, b_t^A - 1)$

(vi) $\bar{\beta}_t^b(b_t^B, b_t^A) \geq \bar{\beta}_t^b(b_t^B + 1, b_t^A - 1)$

At time T, the probability of execution of any limit order is 0. Therefore the conditions hold trivially: $\bar{\beta}_T^b = \underline{\beta}_T^b = \frac{A}{V}$ and $\underline{\beta}_T^s = \bar{\beta}_T^s = \frac{B}{V}$.

Suppose the proposition is false. Then because it is true for T we know $\exists$ a τ such that for $t \geq \tau$, all the elements of the lemma are true, but at τ at least one is false. In what follows we posit the existence of such a τ and show that for each of the possible ensuing states the action is not optimal.

Statement (i) and Statement (iv) must be true at time τ.

Suppose statement (i) is not true at τ, then

$$\underline{\beta}_\tau^s(b_\tau) < \underline{\beta}_\tau^s(b_\tau^B, b_\tau^A + 1).$$

So if $\beta_\tau \in [\underline{\beta}_\tau^s(b_\tau), \underline{\beta}_\tau^s(b_\tau^B, b_\tau^A + 1))$ a seller submits a market order when the book is $(b_\tau^B, b_\tau^A + 1)$ and a limit order when the book is (b_τ^B, b_τ^A). Suppose that the seller instead submits a limit sell order when the book is $(b_\tau^B, b_\tau^A + 1)$, generating a book at time $\tau + 1$ of (b_τ^B, b_τ^A). If the book at time τ is (b_τ^B, b_τ^A), the trader submits a limit order and generates a book at time $\tau + 1$ of $(b_\tau^B, b_\tau^A - 1)$. At time $\tau + 1$, either a buyer arrives or a seller or no one.

a) If a seller arrives then, we know that (i) is true for $t > \tau$ and therefore $p_{\tau+1}^s(b_\tau^B, b_\tau^A - 1) \leq p_{\tau+1}^s(b_\tau^B, b_\tau^A)$. An order submitted at time $\tau + 1$ is only executed if an order submitted earlier has been executed. Therefore, conditional on a seller arriving at time $\tau + 1$,

$$\begin{aligned} &p_\tau^s[(b_\tau^B, b_\tau^A) \mid \text{seller arrives at } \tau + 1] \\ &\quad \leq p_\tau^s[(b_\tau^B, b_\tau^A + 1) \mid \text{seller arrives at time } \tau + 1]. \end{aligned}$$

Therefore

$$\begin{aligned} &[A - \beta_\tau V] p_\tau^s[(b_\tau^B, b_\tau^A + 1) \mid \text{seller at } \tau + 1] \\ &\quad \geq [A - \beta_\tau V] p_\tau^s[b_\tau^B, b_\tau^A) \mid \text{seller at } \tau + 1] \\ &\quad \geq [B - \beta V]. \end{aligned}$$

Hence, conditional on a seller arriving, the payoff to submitting a limit sell order is higher when the book is thinner at time τ. Therefore, conditional on a seller arriving, it cannot be optimal to submit a market order at time τ when the book at the ask is thinner.

b) Suppose instead that a buyer arrives at time $\tau + 1$. Because (v) is true at $\tau + 1$, traders either do not change their behavior or, for

$$\beta_{\tau+1} \in [\bar{\beta}^b_{\tau+1}(b^B_\tau, b^A_\tau); \bar{\beta}^b_{\tau+1}(b^B_\tau, b^A_\tau - 1)],$$

they submit a market order when the book is (b^B_τ, b^A_τ), generating a $\tau + 2$ book of $(b^B_\tau, b^A_\tau + 1)$, and submit a limit order when the book is $(b^B_\tau, b^A_\tau - 1)$, generating a $\tau + 2$ book of $(b^B_\tau + 1, b^A_\tau - 1)$.

Now, for a seller arriving at time $\tau + 2$, because (iii) is true,

$$p^s_{\tau+2}(b^B_\tau, b^A_\tau) \geq p^s_{\tau+2}(b^B_\tau + 1, b^A_\tau - 1).$$

Similarly, because (i) is true,

$$p^s_{\tau+2}(b^B_\tau, b^A_\tau + 1)) \geq p^s_{\tau+2}(b_\tau).$$

But an order submitted at time $\tau + 2$ is only executed if an order submitted at time τ has been executed. Therefore

$$\begin{aligned}
&[A - \beta_\tau V]p^s_\tau[(b^B_\tau, b^A_\tau + 1) \mid \text{buyer arrives at } \tau + 1] \\
&\quad \geq [A - \beta_\tau V]p^s_\tau[(b^B_\tau, b^A_\tau) \mid \text{buyer arrives at } \tau + 1] \\
&\quad \geq [B - \beta_\tau V].
\end{aligned}$$

Hence, conditional on a buyer arriving at time $\tau + 1$, there is a contradiction.

c) If no one arrives then the argument proceeds as per a) and b), but one period ahead.

Statement (i) is therefore true.

A symmetric argument can be constructed for Statement (iv).

Statement (ii) and Statement (v) must be true at time τ.

Suppose Statement (ii) is false so that

$$\underline{\beta}^s_\tau(b^B_\tau, b^A_\tau) < \underline{\beta}^s_\tau(b^B_\tau + 1, b^A_\tau).$$

At time τ for $\beta_\tau \in [\underline{\beta}^s_\tau(b^B_\tau, b^A_\tau), \underline{\beta}^s_\tau(b^B_\tau + 1, b^A_\tau))$ a seller submits a market order if the book is $(b^B_\tau + 1, b^A_\tau)$ and a limit order if the book is (b^B_τ, b^A_τ). Suppose instead that such a seller submitted a limit order in both cases. If the book was (b_τ^B, b_τ^A), it becomes $(b_\tau^B, b_\tau^A - 1)$. Similarly, if it was $(b_\tau^B + 1, b_\tau^A)$, it becomes $(b_\tau^B + 1, b_\tau^A - 1)$.

d) Suppose a buyer arrives at time $\tau + 1$. As (iv) is true, either the buyer is not affected by the thicker book or, for

$$\beta_{\tau+1} \in [\bar{\beta}^b_{\tau+1}(b_\tau^B + 1, b_\tau^A - 1); \bar{\beta}^b_{\tau+1}(b_\tau^B, b_\tau^A - 1)],$$

the buyer will submit a limit order when the book is $(b_\tau^B, b_\tau^A - 1)$ and a market order when the book is $(b_\tau^B + 1, b_\tau^A - 1)$. So if the book at time $\tau + 1$ was $(b_\tau^B, b_\tau^A - 1)$, it becomes $(b_\tau^B + 1, b_\tau^A - 1)$, and if the book at time $\tau + 1$ is $(b_\tau^B + 1, b_\tau^A - 1)$, it becomes $(b_\tau^B + 1, b_\tau^A)$. Suppose a seller arrives at time $\tau + 2$. We know from (i) that

$$p^s_{\tau+2}(b_\tau^B + 1, b_\tau^A)) \geq p^s_{\tau+2}(b_\tau^B + 1, b_\tau^A - 1).$$

But an order placed at time $\tau + 2$ is only executed if an order placed at time τ is executed, because time priority is in effect. Therefore, conditional on a buyer arriving at time $\tau + 1$,

$$\begin{aligned}
&[A - \beta_\tau V]p^s_\tau[(b_\tau^B + 1, b_\tau^A) \mid \text{buyer arrives at } \tau + 1] \\
&\geq [A - \beta_\tau V]p^s_\tau[(b_\tau^B, b_\tau^A) \mid \text{buyer arrives at } \tau + 1] \\
&\geq [B - \beta_\tau V],
\end{aligned}$$

which yields a contradiction.

e) Suppose a seller arrives at time $\tau + 1$. Because (ii) is true at time $\tau + 1$,

$$p^s_{\tau+1}(b_\tau^B, b_\tau^A - 1) \leq p^s_{\tau+1}(b_\tau^B + 1, b_\tau^A - 1).$$

An order submitted at time $\tau + 1$ is only executed if an order submitted at time τ has been executed, therefore

$$\begin{aligned}
[A - \beta_\tau V]p^s_\tau(b_\tau^B + 1, b_\tau^A) \quad &\geq [A - \beta_\tau V]p^s_\tau(b_\tau^B, b^A)) \\
&\geq B - \beta_\tau V
\end{aligned}$$

is a contradiction.

f) Suppose no one arrives at time $\tau + 1$, then the argument proceeds as per d) and e), but one period ahead.

Statement (ii) is therefore true. A symmetric argument can be constructed for Statement (v).

Statement (iii) and Statement (vi) are true at time τ.

Suppose statement (iii) is not true, then

$$\underline{\beta}^s_\tau(b_\tau^B + 1, b_\tau^A - 1) > \underline{\beta}^s_\tau(b_\tau^B, b_\tau^A)$$

Hence, for $\beta_\tau \in [\underline{\beta}^s_\tau(b_\tau^B, b_\tau^A), \underline{\beta}^s_\tau(b_\tau^B + 1, b_\tau^A - 1)]$ a seller will submit a limit order if the book is $(b_\tau^B + 1, b_\tau^A - 1)$ and a market order if the book is (b_τ^B, b_τ^A). Suppose instead that the trader submits a limit order in both cases. If the book is (b_τ^B+1, b_τ^A-1) it generates a book at time $\tau+1$ of (b_τ^B+1, b_τ^A-2), and if the book at time τ is (b_τ^B, b_τ^A) it generates a book at time $\tau+1$ of (b_τ^B, b_τ^A-1).

g) If a buyer arrives at period $\tau+1$, then because (vi) is true at $\tau+1$, for

$$\beta_{\tau+1} \in [\bar{\beta}_{\tau+1}^b(b_\tau^B+1, b_\tau^A-2); \bar{\beta}_{\tau+1}^b(b_\tau^B, b_\tau^A-1]$$

buyers will submit a market order if the book is (b_τ^B+1, b_τ^A-2), generating a time $\tau+2$ book of (b_τ^B+1, b_τ^A-1), and a limit order if the book at time $\tau+1$ is (b_τ^B, b_τ^A-1), generating a book at time $\tau+2$ of (b_τ^B+1, b_τ^A-1). A seller at time $\tau+2$ will therefore face the same book. Therefore

$$\begin{aligned}&[A-\beta_\tau V]p_\tau^s[(b_\tau^B+1, b_\tau^A-1) \mid \text{buyer arrives at } \tau+1]\\ &\quad = [A-\beta_\tau V]p_\tau^s[(b_\tau^B, b_\tau^A) \mid \text{buyer arrives at time } \tau+1],\end{aligned}$$

which yields the contradiction.

h) Suppose a seller arrives at time $\tau+1$, because (iii) is true at time $\tau+1$, this yields a direct contradiction.

i) Suppose no one arrives at time $\tau+1$, then the argument proceeds as per g) and h), but one period ahead.

Statement (iii) is therefore true. A similar argument can be constructed for Statement (vi). It is therefore true. ■

Proof of Proposition 2. The time t probability of observing a transaction at the ask at time $t+1$ is the probability that the time $t+1$ trader market buys

$$\Pr[P_{t+1} = A] = \pi_b[1 - F(\bar{\beta}_{t+1}^b(b_t))].$$

We use the fact that different actions at time t generate different books at time $t+1$. If at time t we observe a transaction at the ask, then the book at time $t+1$ is $b_{t+1} = (b_t^B, b_t^A+1)$. If we observe a transaction at the bid, then the book at time $t+1$ is $b_{t+1} = (b_t^B-1, b_t^A)$. From Proposition 1(vi), we know that

$$\begin{aligned}\bar{\beta}_t^b(b_t^B, b_t^A) &\geq \bar{\beta}_t^b(b_t^B+1, b_t^A-1)\\ \Longrightarrow \bar{\beta}_t^b(b_t^B-1, b_t^A) &\geq \bar{\beta}_t^b(b_t^B, b_t^A-1)\end{aligned}$$

As Proposition 1 applies to all books, from (v) we know that

$$\begin{aligned}\bar{\beta}_t^b(b_t^B, b_t^A) &\leq \bar{\beta}_t^b(b_t^B, b_t^A+1)\\ \Longrightarrow \bar{\beta}_t^b(b_t^B, b_t^A+1) &\leq \bar{\beta}_t^b(b_t^B, b_t^A),\end{aligned}$$

hence

$$\bar{\beta}_t^b(b_t^B-1, b_t^A) \geq \bar{\beta}_t^b(b_t^B, b_t^A+1).$$

Therefore

$$[1 - F(\bar{\beta}_t^b(b_t^B - 1, b_t^A)] \leq [1 - F(\bar{\beta}_t^b(b_t^B, b_t^A + 1)].$$

A symmetric argument can be constructed for the probability of seeing a transaction at the bid, given that the last transaction was at the ask. The result follows. ∎

Proof of Proposition 3. The time t probability of observing a limit buy at time $t+1$ is

$$\Pr[\phi_{t+1}^b(b_{t+1}, \beta_{t+1}) = 1^B] = \pi_b[F(\bar{\beta}_{t+1}^b(b_{t+1})) - F(\underline{\beta}_{t+1}^b(b_{t+1}))].$$

Suppose $b_t^B < T - t - 1$, so that the probability of execution of a limit buy order is not zero, then $\underline{\beta}_{t+1}^b(b_{t+1})$ is independent of the book. If the order at time t was a limit buy order, then the book at time $t+1$ is $(b_t^B + 1, b_t^A)$, if the transaction at time t was not a limit buy order then it was either a market sell so that $b_{t+1} = (b_t^B - 1, b_t^A)$, a limit sell so that $b_{t+1} = (b_t^B, b_t^A - 1)$, a market buy so that $b_{t+1} = (b_t^B, b_t^A + 1)$, or no one arrived at the market so that $b_{t+1} = (b_t^B, b_t^A)$. From Proposition 1(iv) we know that

$$\bar{\beta}_{t+1}^b(b_t^B - 1, b_t^A) \geq \bar{\beta}_{t+1}^b(b_t^B, b_t^A) \geq \bar{\beta}_{t+1}^b(b_t^B + 1, b_t^A).$$

From points (v) and (vi) we know that

$$\bar{\beta}_{t+1}^b(b_t^B, b_t^A) \geq \bar{\beta}_{t+1}^b(b_t^B, b_t^A + 1) \geq \bar{\beta}_{t+1}^b(b_t^B + 1, b_t^A).$$

From point (v) we know that

$$\bar{\beta}_{t+1}^b(b_t^B, b_t^A - 1) \geq \bar{\beta}_{t+1}^b(b_t^B, b_t^A) \geq \bar{\beta}_{t+1}^b(b_t^B + 1, b_t^A).$$

The highest β that corresponds to limit buying $(\bar{\beta}_{t+1}^b(b_t^B + 1, b_t^A))$ is lower than any of the other realizations. $F(\cdot)$ is monotonically nondecreasing in β. $F(\bar{\beta}_{t+1}^b(b_t^B + 1, b_t^A)$ is therefore less than a linear combination of the other realizations.

Suppose now that $|b_t^B| \geq T - t - 1$; that is, either no limit orders are submitted, in which case the proposition holds trivially, or at time t, the extra limit order submitted changes the execution probability to 0. In this case, $\bar{\beta}_{t+1}^b(b_t^B + 1, b_t^A) = \frac{A}{V}$, and the result follows.

A similar argument holds for limit sells. ∎

Proof of Proposition 4. The time t probability of seeing a limit sell order at time $t+1$ is given by

$$\Pr[\phi_{t+1}^s(b_{t+1}, \beta_{t+1}) = -1^A] = \pi_s[F(\bar{\beta}_{t+1}^s(b_{t+1})) - F(\underline{\beta}_{t+1}^s(b_{t+1}))].$$

Suppose $|b_t^A| < T - t - 1$, then $\bar{\beta}_{t+1}^s(b_{t+1})$ is independent of the book.

If the last transaction was at the ask, then the book is thinner at the ask: specifically the book is $(b_t^B, b_t^A + 1)$. If the last transaction was at the bid, then the book is $(b_t^B - 1, b_t^A)$. From Proposition 1, points (ii) and (i) we know that

$$\underline{\beta}^s_{t+1}(b_t^B - 1, b_t^A) \geq \underline{\beta}^s_{t+1}(b_t^B, b_t^A) \geq \underline{\beta}^s_{t+1}(b_t^B, b_t^A + 1),$$

which implies the result.

Now consider the case where $|b_t^A| \geq T - t - 1$. If $|b_t^A| = T - t - 1$, then if there is a transaction at the ask, then $\bar{\beta}_{t+1}$ jumps from $\frac{B}{V}$ to $\frac{A}{V}$, whereas the transaction at the bid does not affect it. Result follows.

A similar argument can be constructed for the probability of observing limit buy transactions. ■

Proof of Proposition 5. The time t probability of seeing a limit sell order at time $t + 1$ is given by

$$\Pr[\phi^s_{t+1}(b_{t+1}, \beta_{t+1}) = -1^A] = \pi_s[F(\bar{\beta}^s_{t+1}(b_{t+1})) - F(\underline{\beta}^s_{t+1}(b_{t+1}))].$$

Suppose $|b_t^A| < T - t - 1$, then $\bar{\beta}^s_{t+1}(b_{t+1})$ is independent of the book. A market buy order at time t yields a book of $(b_t^B, b_t^A + 1)$. A limit buy order at time t yields a book of $(b_t^B + 1, b_t^A)$. A limit sell order at time t yields a book of $(b_t^B, b_t^A - 1)$, while a market sell yields a book of $(b_t^B - 1, b_t^A)$. Proposition 1(iii) states:

$$\underline{\beta}^s_{t+1}(b_t^B, b_t^A) \leq \underline{\beta}^s_{t+1}(b_t^B + 1, b_t^A - 1),$$

which implies both

$$\underline{\beta}^s_{t+1}(b_t^B - 1, b_t^A) \leq \underline{\beta}^s_t(b_t^B, b_t^A - 1)$$

and

$$\underline{\beta}^s_{t+1}(b_t^B, b_t^A + 1) \leq \underline{\beta}^s_t(b_t^B + 1, b_t^A).$$

Using the fact from Proposition 1(ii) that

$$\underline{\beta}^s_{t+1}(b_t^B, b_t^A) \geq \underline{\beta}^s_{t+1}(b_t^B + 1, b_t^A)$$

yields

$$\underline{\beta}^s_{t+1}(b_t^B, b_t^A + 1) \leq \underline{\beta}^s_t(b_t^B + 1, b_t^A) \leq \underline{\beta}^s_{t+1}(b_t^B - 1, b_t^A) \leq \underline{\beta}^s_t(b_t^B, b_t^A - 1),$$

which implies the result.

Now consider the case where $|b_t^A| \geq T - t - 1$. If $|b_t^A| = T - t - 1$, then if there is a transaction at the ask, then $\bar{\beta}_{t+1}$ jumps from $\frac{B}{V}$ to $\frac{A}{V}$, whereas the transaction at the bid does not affect it. Result follows.

A similar argument can be constructed for the probability of limit buy transactions and other orders. ∎

Proof of Proposition 6. If there are two ticks, then the probability that the ask rises after the bid has dropped is 0. Hence the probability that the ask drops after the bid has dropped is higher than the probability that the ask increases or remains the same. ∎

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