MBA201A - Monopoly Pricing Handout

Ed Egan

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Demand

Demand is a function that describes how much of a good, q, a consumer wants given a price p. Inverse demand is the opposite: how much would a consumer be willing to pay, p, given a quantity of a good q.

$$\text{Demand:} \quad q = \frac{A}{B} - \frac{1}{B} \cdot p$$
$$\text{Inverse Demand:} \quad p = A - B \cdot q$$

Note that the demand function could be written as $q = X + Y \cdot p$, but the same variables (A and B) have been used from the inverse demand function instead. You can rearrange demand to get inverse demand, and vice versa.

When plotting a graph the price p goes on the y-axis (the vertical axis) and the quantity q goes on the x-axis (the horizontal axis). Therefore we plot the inverse demand function on graphs. **Inverse demand slopes down**. This is always true - that is why the equation is written with a minus sign before the slope (B). There is no (undisputed) evidence of Giffen goods actually existing.

There are two easy ways of doing the plot of the inverse demand function $p = A - B \cdot q$:

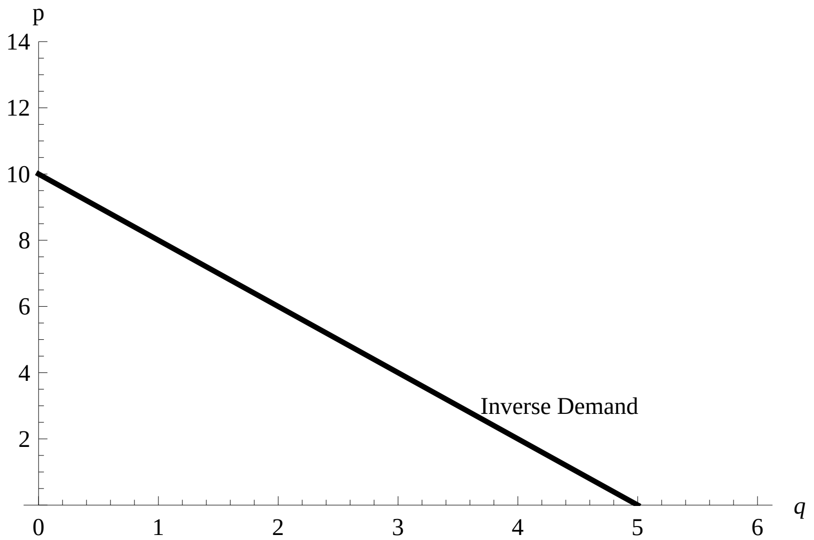
1. A is the y-intercept, and $-B$ is the slope (i.e. $\frac{dy}{dx}$), so start from A and then go B units **down** for every unit that you go right.
2. Put a point at $(0, A)$ and another at $(\frac{A}{B}, 0)$ (set $p = 0$ and solve the inverse demand function, or the equivalent demand function), and then join the points with a line.

Note that coordinates on a plot are given as (x, y) by convention, which for us is always (q, p).

In our models demand will only be a function of price. However, in the real world (and more complex models in economics), demand is a function of many things, including:

- Income
- Availability of substitutes
- Taste
- Demographics

Furthermore, in our models demand is strictly continuous. That is for every price, even those including tiny fractions of a cent, you have a demand for a quantity and this demand may be for a tiny fraction of a unit. This allows us to use calculus (differentiation) to find answers. In the real world, demand may not be continuous.



In the graph above, the inverse demand curve $p = 10 - 2q$ has been used. Note the slope of -2 (which slopes down), and the y-intercept of 10, the x-intercept of $\frac{10}{2} = 5$.
No matter whether you are maximizing profit as a monopolist, pricing in a competitive market, maximizing revenue, producing at a capacity quantity, or setting an arbitrary price, or in fact making any kind of pricing or quantity decision as a firm, your **price-quantity pair must lie on the (inverse) demand curve**. Price-quantity pairs that lie anywhere other than on the demand curve are impossible.

Aggregating Demand

When you aggregate demand you must **sum demand functions** (over the correct ranges). You must **NOT** sum inverse demand functions.
Suppose you are given two groups X and Y, with N_X people in group X and N_Y people in group Y, and the following demand functions:

$$q_X = \frac{A_X}{B_X} - \frac{1}{B_X} \cdot p$$

$$q_Y = \frac{A_Y}{B_Y} - \frac{1}{B_Y} \cdot p$$

The over the range where both demand functions are valid, which is discussed below, the total demand Q is then:

$$Q = N_X \cdot q_X + N_Y \cdot q_Y$$

$$\therefore Q = N_X \cdot \left(\frac{A_X}{B_X} - \frac{1}{B_X} \cdot p\right) + N_Y \cdot \left(\frac{A_Y}{B_Y} - \frac{1}{B_Y} \cdot p\right)$$

But what about these 'relevant ranges'? Rearranging the demand functions to get inverse demand functions we have:

$$p = A_X - B_X \cdot q_X$$

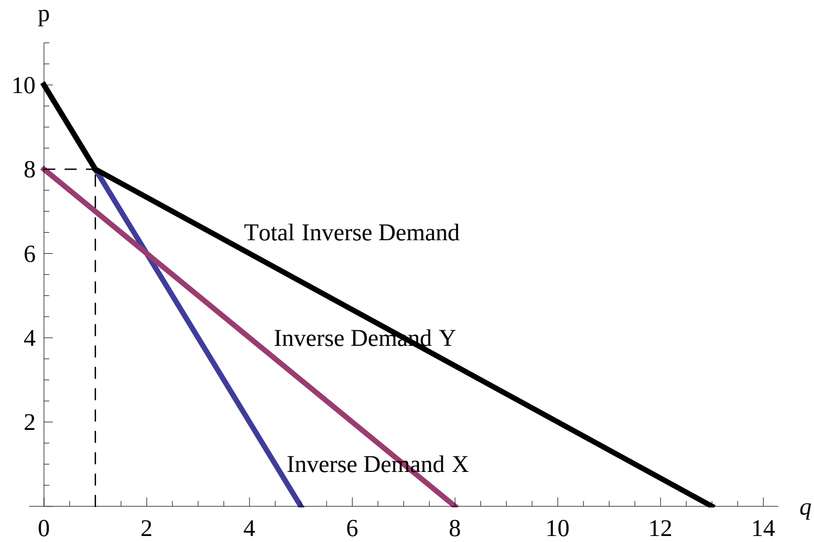
$$p = A_Y - B_Y \cdot q_Y$$

Suppose that $A_X > A_Y$, then at $p = A_X$ there are no consumers from group Y that want to buy - the price is greater than their willingness to pay for even the tiniest fraction of a unit. In this case, when $p \in (\infty, A_X]$ (this notation says p is in the range from but not including infinity to and including A_X) there is no demand, when $p \in (A_X, A_Y]$ there is only demand from group X, and from $p \in (A_Y, 0]$ both groups have demand and we sum them as above.

In practice you will want to convert these p ranges to q ranges. You do this by plugging p in to the right demand function. To find the range of q that will hold when $p \in (A_X, A_Y]$, you take both limits of the range and plug them into the inverse demand for the X group, as this is the only function that is valid then. Of course, the first answer will be zero ($p = A_x = A_x - B_x \cdot q \implies B_x \cdot q = 0$). Likewise for $p \in (A_Y, 0]$ the aggregage demand function is valid. However, you have already solved the quantity where $p = A_Y$ in the previous calculation and there is no need to repeated. Therefore:

$$Q = \begin{cases} q_X & \text{when } q \in \left(0, \frac{A_x - A_y}{B_x}\right] \\ N_X \cdot q_X + N_Y \cdot q_Y & \text{when } q \in \left(\frac{A_x - A_y}{B_x}, \frac{A_{Total}}{B_{Total}}\right] \end{cases}$$

Now let's practice this with an example!



In the graph above, group X's inverse demand curve has an equation $p = 10-2q$, and group Y's inverse demand curve has an equation $p = 8-q$. Up until a price of 8 only the first demand curve is active. Plugging $p = 8$ into group X's demand curve gives a quantity at which both demand curves are active ($q = 1$).This is the same as doing $q = \frac{10-8}{2} = 1$, using the formula above.

We then create the aggregate demand curve. To do this we must use demand and not inverse demand. Therefore we have:

$$p_X = 10 - 2q \implies q_X = 5 - \frac{p}{2}$$
$$p_Y = 8 - q \implies q_Y = 8 - p$$

We will assume that there there is one person for each group, that is $N_1 = 1$ and $N_2 = 1$. If you are given demand curves for two groups then you also add them in this way.

Recalling that:

$$Q = N_X \cdot q_X + N_Y \cdot q_Y$$

We have:

$$Q = 1 \cdot \left(5 - \frac{p}{2}\right) + 1 \cdot (q_2 = 8 - p) = 13 - \frac{3p}{2}$$
$$\therefore P = \frac{26}{3} - \frac{2Q}{3}$$

To determine the where this new aggregate demand curve hits the x-axis, plug $P = 0$ into the aggregate demand function to get a value of $Q = 13$.

Thus our aggregate demand function is:

$$Q = \begin{cases} 5 - \frac{p}{2} & \text{when } q \in (0, 1] \\ 13 - \frac{3p}{2} & \text{when } q \in (1, 13] \end{cases}$$

And our aggregate inverse demand function is the inverse of the respective components:

$$P = \begin{cases} 10 - 2Q & \text{when } q \in (0, 1] \\ \frac{26}{3} - \frac{2Q}{3} & \text{when } q \in (1, 13] \end{cases}$$

Maximizing Revenue

Revenue is the number of units sold, q, multiplied by the price p. Recall that price is a function of quantity (and quantity is a function of price). The inverse demand function states the value of price in terms of the quantity demanded. Therefore we have:

$$\text{Revenue:} \quad R = p \cdot q = p(q) \cdot q = (A - Bq) \cdot q = Aq - Bq^2$$

There are three mathematically equivalent ways of solving for marginal revenue. Marginal revenue is the revenue earned from each additional unit sold. If we want to maximize revenue, we want to stop producing at the level where the revenue earned from the next unit is zero. This logic is true for all functions - to solve for a maximum set the first order condition to zero (and check the second order condition is negative, which indicates a maximum rather than a minimum or a point of inflection). The first order condition is the first dervitive with respect to quantity - as we want the quantity at which to stop producing.

To differentiate something the rule is: **Bring down the power and then knock one off the power**. So the derivitive of $y = 1 + 2x + 3x^2$, for example is $\frac{dy}{dx} = 2 + 6x$.

Therefore our options are:

1. Differentiate revenue with respect to quantity using the chain rule and substitute in:

$$\frac{d}{dq}(R) = \frac{d}{dq}(p(q) \cdot q) = \frac{dp(q)}{dq} \cdot q + \frac{dq}{dq} \cdot p$$

$$\frac{d(A - Bq)}{dq} \cdot q + 1 \cdot (A - Bq) = -B \cdot q + (A - Bq) = A - 2Bq$$

2. Substitute in the inverse demand function and then differentiate:

$$\frac{d}{dq}(R) = \frac{d}{dq}(p(q) \cdot q) = \frac{d}{dq}((A - Bq) \cdot q)$$

$$= \frac{d}{dq}\left((Aq - Bq^2)\right) = A - 2Bq$$

3. Use the rule that marginal revenue has twice the slope of revenue and the same intercept (this rule works with all linear functions):

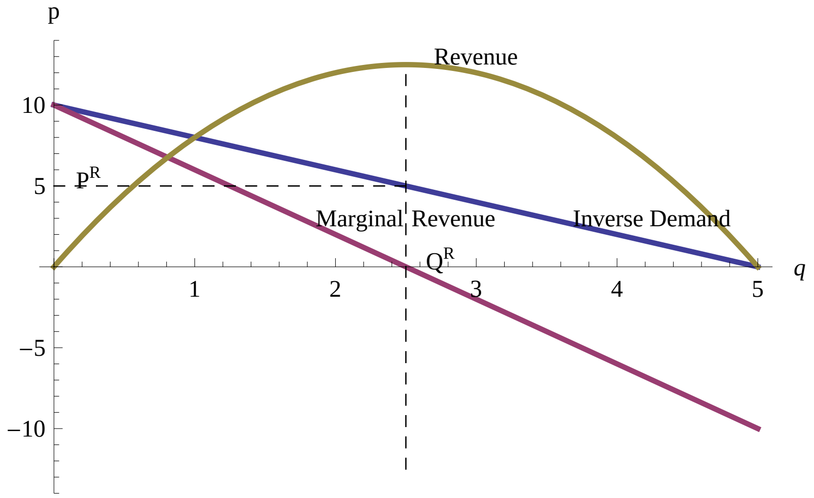
$$\frac{d}{dq}\left(R\right) = A - 2 \cdot Bq$$

Various notation is used. The following things are all the same:

$$\frac{d}{dq}\left(R\right) = R' = MR$$

Again, to find the maximum of a function with respect to a variable, differentiate with respect to that variable and set the result (the first order condition) equal to zero. Therefore to find the quantity that maximizes revenue we do:

$$\frac{d}{dq}\left(R\right) = A - 2 \cdot Bq = 0 \quad \therefore q = \frac{A}{2B}$$



In the diagram above, the inverse demand curve $p = 10 - 2q$ has been used again. Recall that the marginal revenue curve has twice the slope and so is $MR = 10 - 4q$, which intercepts the x axis at $\frac{A}{2B} = \frac{10}{2 \cdot 2} = 2.5$. The total revenue curve has also be plotted. It is given by $R = p \cdot q = 10q - 2q^2$. The peak of the revenue curve is at the point where the marginal revenue equals zero. This is given by q^R on the diagram. Beyond that point marginal revenue is negative, so we would not want to produce an additional unit, even if the cost of production were zero. To find the price you should charge to maximize revenue, you plug the revenue maximizing quantity into the (inverse) demand curve - this is indicated by the dotted lines and gives a price of p^R.

Elasticity of Demand

Demand curves have 'elasticities'. By convention this is denoted by a Greek epsilon: ϵ. The linear and downward sloping inverse demand curves that you will see have different elasticities at different points. There are some other special demand curves that have constant elasticities. The elasticity of a demand curve is defined as the percentage change in quantity divided by the percentage change in price. You might see it expressed as any of the following (they are all equivalent, at least if demand is continuous):

$$\epsilon = \frac{\%\Delta q}{\%\Delta p} = \frac{\frac{\Delta q}{q}}{\frac{\Delta p}{p}} = \frac{\Delta q}{\Delta p}\frac{p}{q} = \frac{dq}{dp}\frac{p}{q} = \frac{1}{\frac{dp}{dq}}\frac{p}{q}$$

By taking the inverse demand function we can calculate a formula for the elasticity of our demand curve:

$$\epsilon = \frac{1}{\frac{dp}{dq}} \cdot \frac{p}{q} = \frac{1}{\frac{d(A-Bq)}{dq}}\frac{(A-Bq)}{q} = -\frac{(A-Bq)}{Bq} = 1 - \frac{A}{Bq}$$

Now for any quantity we can quickly calculate the elasticity. The elasticity is always negative and is divided into three ranges:

- Elastic: $\epsilon \in (-\infty, 1)$
- Unitary: $\epsilon = 1$
- Inelastic: $\epsilon \in (1, 0)$

These ranges run from left to right on a standard diagram. Let's check this with three points:

- Elastic ($q = 0$):
  We can't plug in $q = 0$ to the equation above, but you should be able to see that as q approaches 0, ϵ approaches negative infinity. Formally:

  $$\lim_{q \to 0}(1 - \frac{A}{Bq}) = -\infty$$

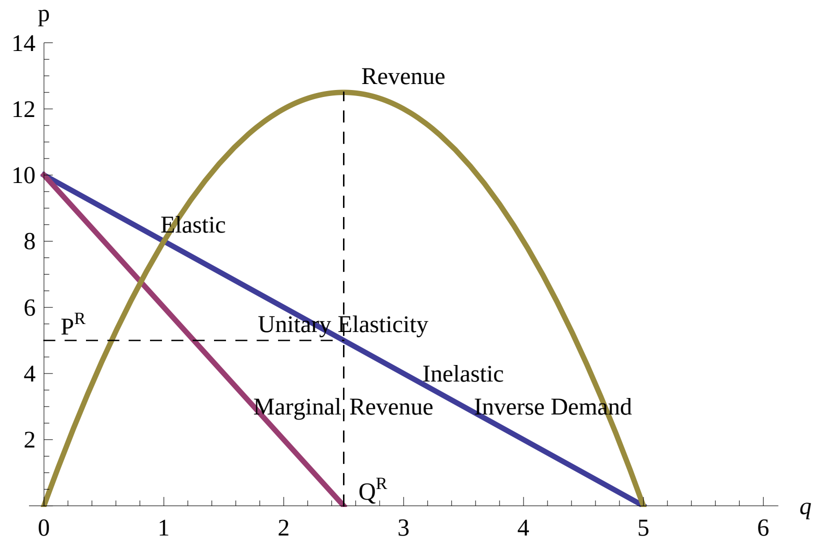
- Unitary: (q maximizes revenue):
  The two portions of the demand curve are divided in to the elastic part, where increasing quantity will increase revenue, and the inelastic part, where increasing quantity will decrease revenue. The dividing point is the point where revenue is maximized, which has an elasticity of one. This can be seen as follows:

  $$\frac{d}{dq}(R) = A - 2 \cdot Bq = 0 \quad \therefore q = \frac{A}{2B}$$

  $$\epsilon = 1 - \frac{A}{Bq} = 1 - \frac{A}{B \cdot \left(\frac{A}{2B}\right)} = 1 - 2 = -1$$

- Inelastic: ($p = 0$ or $q = \frac{A}{B}$):
  The point where the inverse demand curve reaches $p = 0$ is given by $q = \frac{A}{B}$. Plugging this value in gives an elasticity of zero:

$$\epsilon = 1 - \frac{A}{Bq} = 1 - \frac{A}{B \cdot \left(\frac{A}{B}\right)} = 1 - 1 = 0$$



In the diagram above, the inverse demand curve $p = 10 - 2q$ has been used yet again. The point where revenue is maximized (given by either the peak of the R curve, or the point where $MR = 0$) is the point of unitary elasticity on the demand curve. For higher prices (or lower quantities, as the price quantity pair must lie on the demand curve) you are on the elastic portion of the demand curve. On this portion an additional unit of production will bring in more revenue (though it may not outweigh the cost - which is discussed next). For prices that are lower than the revenue maximizing price p^R, you are on the inelastic portion of the demand curve. On this portion of the demand curve each additional unit produced will reduce your revenue (MR is negative beyond this point). Thus, irrespective of your costs of production, this can not be optimal. You should immediately raise your price (or curtail your output) to get back to the elastic portion of the demand curve.

Monopoly

A monopolist faces no competition in the market (you should want to become a monopolist!), and so produces until the marginal revenue of an additional unit is equal to the marginal cost of producing that unit. This is because the monopolist just wants to maximize profit. We will use the notation C to indicate total cost. If cost is linear then $C = F + Dq$, where F is the fixed cost and D is the marginal cost. If you are given a more complex cost function then you will have to determine marginal cost by taking the derivitive of total cost with respect to quantity.

$$\text{Profit} = \pi = R - C$$

$$\max \pi \implies \frac{d}{dq}(R) - \frac{d}{dq}(C) = 0 \quad \frac{d}{dq}(R) = \frac{d}{dq}(C)$$

We have already discussed marginal revenue and you given a cost function you can determine marginal cost. Suppose the cost function is linear as above, then marginal cost is given by $\frac{d}{dq}(C) = \frac{d}{dq}(F + Dq) = D$. So linear cost functions give constant marginal costs.

Now we can solve for a general solution for the profit maximizing quantity for a monopolist for linear inverse demand and linear total costs:

$$MR = MC$$

$$\therefore A - 2Bq = D \quad \therefore q = \frac{A - D}{2B}$$

Notice that the profit maximizing quantity is not affected by fixed costs. Now to solve for the profit maximizing price and profit we plug this answer back in to the inverse demand function and the profit function:

$$p = A - Bq \quad \therefore p = A - B\left(\frac{A - D}{2B}\right) = \frac{A - D}{2}$$

$$\pi = R - C = p \cdot q - (F + Dq) = \left(\frac{A - D}{2}\right) \cdot \left(\frac{A - D}{2B}\right) - F - D\left(\frac{A - D}{2B}\right)$$

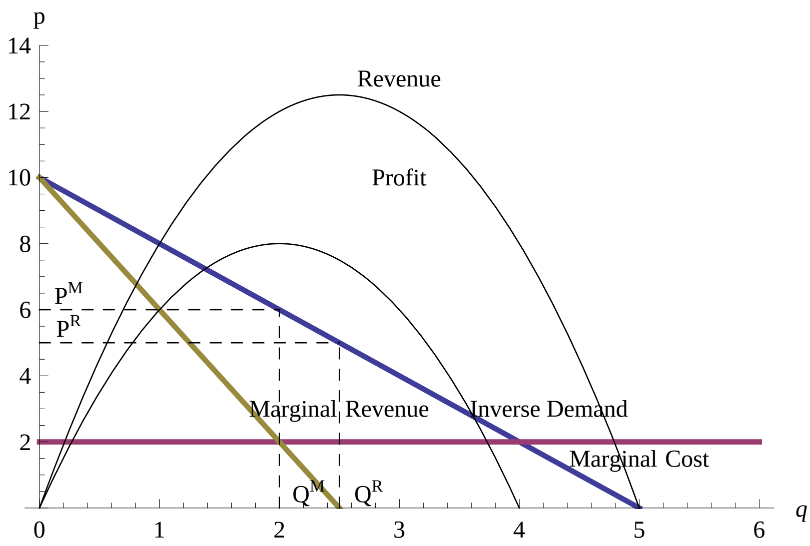
Althought the above equation does not look very friendly, you should notice that there are no p or q terms in it, and in fact it is entirely composed of parameters that will be given to you: A, B, D, F.

The monopolist only uses the fixed costs, F, to calculate profit. If profit is not positive then the monopolist should immediately cease production altogether.

Finally, you should note that the monopoly quantity $q = \frac{A-D}{2B}$ can be plugged back into the elasticity equation from before. When this is done we find that:

$$\epsilon = -\frac{A + D}{A - D}$$

which is more negative (i.e. on the elastic portion of the demand curve) for all marginal costs greater than zero. Therefore monopolists, in maximizing profit, produce less than they would if they wanted to maximize revenues.



In the diagram above, the inverse demand curve $p = 10 - 2q$ has been used yet again, but two new curves have been added: A linear marginal cost curve with $MC = 2$ gives the cost of producing additional units, and a total profit curve ($\pi = p \cdot q - (F + Dq) = p \cdot q - 2q$), which assumes the fixed cost is zero ($F = 0$). You should be able to see from the diagram that profits are maximized when $MR = MC$, and that this gives a q^m which is less than q^R, meaning that profit maximization automatically implies that you are on the elastic portion of the demand curve. To determine the profit maximizing price you take the q^M that is given by $MR = MC$ and you plug it into the demand curve. This will give you p^M on the diagram, which you should see is larger than p^R.