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## The Society for Financial Studies

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Source: *The Review of Financial Studies*, Vol. 16, No. 2 (Summer, 2003), pp. 301-343

Published by: [Oxford University Press](#). Sponsor: [The Society for Financial Studies](#).

Stable URL: <http://www.jstor.org/stable/1262678>

Accessed: 24/01/2011 20:04

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# Liquidity-Based Competition for Order Flow

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We present a microstructure model of competition for order flow between exchanges based on liquidity provision. We find that neither a pure limit order market (PLM) nor a hybrid specialist/limit order market (HM) structure is competition-proof. A PLM can *always* be supported in equilibrium as the dominant market (i.e., where the hybrid limit book is empty), but an HM can also be supported, for some market parameterizations, as the dominant market. We also show the possible coexistence of competing markets. Order preferencing—that is, decisions about where orders are routed when investors are indifferent—is a key determinant of market viability. Welfare comparisons show that competition between exchanges can *increase* as well as reduce the cost of liquidity.

Active competition between exchanges for order flow in cross-listed securities is intense in the current financial marketplace. Examples include rivalries between the New York Stock Exchange (NYSE), crossing networks, and ECNs and between the London Stock Exchange, the Paris Bourse, and other continental markets for equity trading and between Eurex and London International Financial Futures and Options Exchange (LIFFE) for futures volume. While exchanges compete along many dimensions (e.g., “payment for order flow,” transparency, execution speed), liquidity and “price improvement” will, in our view, be the key variables driving competition in the future. Over time, high-cost markets should be driven out of business as investors switch to cheaper trading venues. Moreover, “market structure” is increasingly singled out by regulators, exchanges, and other market participants as a major determinant of liquidity.<sup>1</sup>

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We thank the editor, Larry Glosten, for many helpful insights and suggestions. We also benefited from comments from Shmuel Baruch, Utpal Bhattacharya, Bruno Biais, Wolfgang Bühler, David Goldreich, Rick Green, Burton Hollifield, Ronen Israel, Craig MacKinlay, Uday Rajan, Robert Schwartz, George Sofianos, Tom Tallarini, Jr., Josef Zechner, as well as from seminar participants at the Catholic University of Louvain, London Business School, Mannheim University, Stockholm School of Economics, Tilburg University, University of Utah, University of Vienna, Wharton School, and participants at the 1997 WFA and 1997 EFA meetings and the 1999 RFS Price Formation conference in Toulouse. Financial support from the University of Vienna during Seppi's 1997 sabbatical is gratefully acknowledged. Address correspondence to: Duane Seppi, Graduate School of Industrial Administration, Carnegie Mellon University, Pittsburgh, PA 15213-3890, or e-mail: ds64@andrew.cmu.edu.

<sup>1</sup> See Levitt (2000) and NYSE (2000) regarding the U.S. equity market and “One World, How Many Stock Exchanges?” in the *Wall Street Journal*, May 15, 2000, Section C, page 1, for a summary of developments in the global equity market. See also LIFFE (1998).

The coexistence of competing markets raises a number of questions. Do liquidity and trading naturally concentrate in a single market? Is the current upheaval simply a transition to a new centralized trading arrangement? Or will competing markets continue to coexist side by side in the future? If multiple exchanges can coexist, is the resulting fragmentation of order flow desirable from a policy point of view? Do some market designs provide inherently greater liquidity than others on particular trade sizes?<sup>2</sup> If so, which types of investors prefer which types of markets? If not, do the observed differences in liquidity simply follow from locational cost advantages (e.g., is the Frankfurt-based Eurex the natural “dominant” market for Bundt futures)? Is there a constructive role for regulatory policy in enhancing market liquidity?

To answer such questions the economics of both liquidity supply and demand must be understood. In this article we study competition between two common market structures. The first is an “order driven” pure limit order market in which investors post price-contingent orders to buy/sell at preset limit prices. The Paris Bourse and ECNs such as Island are examples of this structure. The second is a hybrid structure with both a specialist and a limit book. The NYSE is the most prominent example of this type of market.

Limit orders and specialists, we argue, play central roles in the supply of liquidity. However, there is a timing difference which is key to modeling and understanding these two types of liquidity provision. Limit orders, in either a pure or a hybrid market, represent *ex ante* precommitments to provide liquidity to market orders which may arrive sometime in the future. In contrast, a specialist provides supplementary liquidity through *ex post* price improvement *after* a market order has arrived. A pure limit order market has only the first type of liquidity provision, whereas a hybrid market has both. This difference in the form of liquidity provision, in turn, plays an important role in the outcome of competition between these two types of markets.

In this article we adapt the limit order model of Seppi (1997) to investigate interexchange competition for order flow.<sup>3</sup> In particular, we jointly model both liquidity demand (via market orders) and liquidity supply (via limit orders, the specialist, etc.). Briefly, this is a single-period model in which limit orders are first submitted by competitive value traders (who do not need to trade *per se*) to the two rival markets. An active trader then arrives

<sup>2</sup> Blume and Goldstein (1992), Lee (1993), Peterson and Fialkowski (1994), Lee and Myers (1995), and Barclay, Hendershott, and McCormick (2001) find significant price impact differences of several cents across different U.S. markets. For international evidence see de Jong, Nijman, and Röell (1995) and Frino and McCorry (1995).

<sup>3</sup> Other equilibrium models of limit orders, with and without specialists, are in Byrne (1993), Glosten (1994), Kumar and Seppi (1994), Chakravarty and Holden (1995), Rock (1996), Parlour (1998), Foucault (1999), Viswanathan and Wang (1999), and Biais, Martimort, and Rochet (2000). Cohen et al. (1981), Angel (1992), and Harris (1994) describe optimal limit order strategies in partial equilibrium settings. In addition, Biais, Hillion, and Spatt (1995), Greene (1996), Handa and Schwartz (1996), Harris and Hasbrouck (1996), and Kavajecz (1999) describe the basic empirical properties of limit orders and Hollifield, Miller, and Sandas (2002) and Sandas (2001) carry out structural estimations.

and submits market orders. In the pure market, the limit and market orders are then mechanically crossed, while in the hybrid market, they are executed with the intervention of a strategic specialist. As a way of minimizing her total cost of trading, the active trader can split her orders between the two competing exchanges. Limit order execution is governed by local price, public order, and time priority rules on each exchange. Order submission costs are symmetric across markets. This lets us assess the competitive viability of different microstructures on a “level playing field.”<sup>4</sup>

Order splitting between markets appears in two guises in our article. The first is cost-minimizing splits which strictly reduce the active investor’s trading costs. These involve trade-offs between equalizing marginal prices across competing limit order books and avoiding discontinuities in the specialist’s pricing strategy. The second type of order splitting is a “tie-breaking” rule used when the cost-minimizing split between the two markets is not unique. This second type of splitting—which we call *order preferencing*—is controversial. For example, the ability of brokers on the Nasdaq to direct order flow to the dealer of their choice so long as the best prevailing quote is matched (i.e., to ignore time priority) has been criticized as potentially collusive. Similarly the NYSE is critical of the ability of retail brokers to direct customer orders to regional markets so long as the NYSE quotes are matched.<sup>5</sup> Our analysis below shows that concerns about order preferencing are well founded since “tie-breaking” rules play a key role in equilibrium selection.

Our analysis follows the lead of Glosten (1994) in that we study the optimal design of markets in terms of their competitive viability. In his article Glosten specifically argues that a pure limit order market is competition-proof in the sense that rival markets earn negative expected profits when competing against an equilibrium pure limit order book. We show, however, that multiple equilibria exist if liquidity providers have heterogeneous costs. In some of these equilibria the competing exchanges can coexist, while in others the hybrid market may actually dominate the pure limit order market. Our main results are

- Multiple equilibria can be supported by different preferencing rules. Neither the pure limit order market nor the hybrid market is exclusively competition-proof.
- Competition between exchanges—as new markets open or as firms cross-list their stock—can increase or *decrease* aggregate liquidity relative to a single market environment.

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<sup>4</sup> While actual order submission costs may still differ across exchanges, technological innovation and falling regulatory barriers have dramatically reduced the scope of any natural (i.e., captive) investor clienteles.

<sup>5</sup> Much of the controversy revolves around the possibility of forgone price improvement due to unposted liquidity *inside* the NYSE spread. However, even when all unposted liquidity is optimally exploited, order preferencing still has a significant impact on intermarket competition in our model.

- “Best execution” regulations limiting intermarket price differences to one tick greatly improve the competitive viability of a hybrid market relative to a pure limit order market.

A few other articles also look at competition between exchanges. The work most closely related to ours is Glosten (1998), which looks at competition with multiple pure limit order markets and different precedence rules. Hendershott and Mendelson (2000) model competition between call markets and dealer markets. Santos and Scheinkman (2001) study competition in margin requirements and Foucault and Parlour (2000) look at competition in listing fees. Otherwise, market research has largely taken a regulatory approach in which the pros and cons of different possible structures for a single market are contrasted. Glosten (1989) shows that monopolistic market making is more robust than competitive markets to extreme adverse selection. Madhavan (1992) finds that periodic batch markets are viable when continuous markets would close. Biais (1993) shows that spreads are more volatile in centralized markets (i.e., exchanges) than in fragmented markets (e.g., over-the-counter [OTC] telephone markets). Seppi (1997) finds that large institutional and small retail investors get better execution on hybrid markets, while investors trading intermediate-size orders may prefer a pure limit order market. His result suggests that competing exchanges may cater to specific order size clienteles. Viswanathan and Wang (2002) contrast pure and hybrid market equilibria with risk-averse market makers.

This article is organized as follows. Section 1 describes the basic model of competition between a pure limit order market and a hybrid specialist/limit order market, and Section 2 presents our results. Section 3 compares trading and liquidity across other institutional arrangements. Section 4 summarizes our findings. All proofs are in the appendix.

## 1. Competition Between Pure and Hybrid Markets

We consider a liquidity provision game along the lines of Seppi (1997) in which two exchanges—a pure limit order market (PLM) and a hybrid market (HM) with both a specialist and a limit order book—compete for order flow. In the model, both the supply and demand for liquidity in each market are endogenous. A timeline of events is shown in Figure 1.

Liquidity is demanded by an *active trader* who arrives at time 2 and submits market orders to the two exchanges. The total number of shares  $x$  which she trades is random and exogenous. With probability  $\alpha$  she wants to buy and with probability  $1 - \alpha$  she must sell. The distribution over the random (unsigned) volume  $|x|$  is a continuous strictly increasing function  $F$ . Since the model is symmetric, we focus expositionally on trading when she must buy  $x > 0$  shares. As in Bernhardt and Hughson (1997), the active trader minimizes her total trading cost by splitting her order across the two markets. In particular, let  $B^h$  denote the number of shares she sends as a market

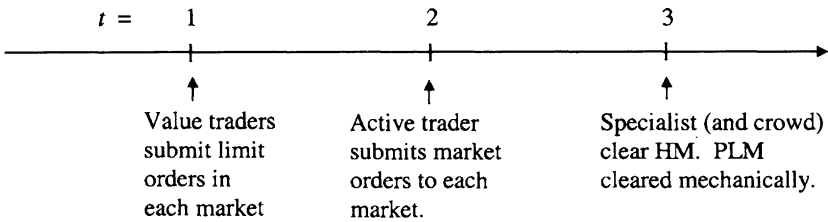


Figure 1  
Timeline for sequence of events

buy to the hybrid market and let  $B^p = x - B^h$  be the market buy sent to the pure market.

Liquidity is supplied by three types of investors. At time 1, competitive risk-neutral *value traders* post limit orders in the pure and hybrid markets' respective limit order books. At time 3, additional liquidity is provided by *trading crowds*—competitive groups of dealers who stand ready to trade whenever the profit in either market exceeds a hurdle level  $r$ . In addition, a single strategic *specialist* with a cost advantage over both the value traders and the crowd provides further liquidity on the hybrid market. All of the liquidity providers have a common valuation  $v$  for the traded stock. Thus the main issue is how much of a premium over  $v$  the active trader must pay for immediacy so as to execute her trades.

Collectively the actions of the various liquidity providers—described in greater detail below—lead to competing liquidity supply schedules,  $T^h$  and  $T^p$ , in the two exchanges. In particular,  $T^h(B^h)$  is the cost of liquidity in the hybrid market when buying  $B^h$  shares (i.e., the premium in excess of the shares' underlying value  $vB^h$ ) and  $T^p(B^p)$  is the corresponding price of liquidity in the pure limit order market. Given the two liquidity supply schedules and the total number of shares  $x$  to be bought, the active trader chooses market orders,  $B^h$  and  $B^p$ , to minimize her trading costs:

$$\min_{B^h, B^p \text{ s.t. } B^h + B^p = x} T^h(B^h) + T^p(B^p). \quad (1)$$

Solving the active trader's optimization [Problem (1)] for each possible volume  $x > 0$  lets us construct order submission policy functions,  $B^p(x)$  and  $B^h(x)$ . These two policy functions, together with the distribution  $F$  over  $x$ , induce endogenous probability distributions  $F^p$  and  $F^h$  over the arriving market orders  $B^p$  and  $B^h$  in the pure and hybrid markets and, hence, over the random payoffs to liquidity providers. In equilibrium, the demand for liquidity in the two markets, as given by  $F^p$  and  $F^h$ , and the liquidity supply schedules,  $T^p$  and  $T^h$ , must be consistent with each other. One goal of this article is to describe the equilibrium relation between the market order arrival distributions and the liquidity supply schedules. What types of

market orders are sent to which markets? What do the limit order books and liquidity supply schedules look like? How do regulatory linkages between the two markets affect trading and liquidity provision? With this overview, we now describe the model in greater detail.

### 1.1 Market environment

For simplicity, prices in both exchanges are assumed to lie on a common discrete grid  $\mathcal{P} = \{\dots, p_{-1}, p_1, p_2, \dots\}$ . Prices are indexed by their ordinal position above or below  $v$ , the liquidity providers' current common valuation of the stock. By taking  $v$  to be a constant, we abstract from the price discovery/information aggregation function of markets and focus solely on their liquidity provision role. Like Seppi (1997), this is a model of the transitory (rather than the permanent) component of prices.<sup>6</sup> If  $v$  itself is on  $\mathcal{P}$ , then it is indexed as  $p_0$ . Since the active investor is willing to trade at a discount/premium to  $v$  to achieve immediacy, she must have a private valuation differing from  $v$ .

### 1.2 Limit orders and order execution mechanics

Limit orders play a central role—in our model as well as in actual markets—both by providing liquidity directly and by inducing the hybrid market specialist to offer price improvement. Let  $S_1^h, S_2^h, \dots$  denote the total limit sells posted at prices  $p_1, p_2, \dots$  in the hybrid market and let  $Q_j^h = \sum_{i=1}^j S_i^h$  be the corresponding cumulative depths at or below  $p_j$ . Define  $S_1^p, S_2^p, \dots$  and  $Q_j^p$  similarly for the pure market. All order quantities are unsigned (nonnegative) volumes.

Investors incur up-front submission costs of  $c_j$  per share when submitting limit orders at price  $p_j$ . We interpret these costs—which are ordered  $c_1 > c_2 > \dots$  at  $p_1, p_2, \dots$ —as a reduced form for any costs borne by investors who precommit ex ante to provide liquidity such as, for example, the risk of having their limit orders adversely “picked off” [see Copeland and Galai (1983)].

Limit orders are protected by local priority rules in each exchange. In the pure limit order market, *price priority* requires that all limit sells at prices  $p_j < p$  must be filled before any limit sells at  $p$  are executed. Given price priority, a market buy  $B^p$  is mechanically crossed against progressively higher limit orders in the PLM book until a *stop-out* price  $p^p$  is reached. When executed, limit sells trade at their posted limit prices which may be less than  $p^p$ . At the stop-out price, *time priority* stipulates that if the available limit and crowd orders at  $p^p$  exceed the remaining (unexecuted) portion of  $B^p$ , then they are executed sequentially in order of submission time.

<sup>6</sup> See Stoll (1989), Hasbrouck (1991, 1993), and Huang and Stoll (1997). Seppi (1997) shows that his analysis carries over in a single market setting if  $v$  is a function of the arriving market orders, but that the algebraic details are more cumbersome.

The hybrid market has its own local priority rules. When a market order  $B^h$  arrives in the hybrid market, the specialist sets a *cleanup* price  $p^h$  at which he clears the market on his own account after first executing any orders with priority. In addition to respecting time and price priority, the specialist is also required by *public priority* to offer a better price than is available from the unexecuted limit orders in the HM book or from the crowd. Thus, to trade himself, the specialist must undercut both the crowd and the remaining (unexecuted) HM limit order book.

The priority rules are local in that each exchange's rules apply only to orders on that exchange. The pure market is under no obligation to respect the priority of limit orders in the hybrid book and vice versa. Priority rules which apply globally across exchanges create, in effect, a single integrated market. Section 3 explores the impact of cross-market priority rules.

### 1.3 The trading crowd

As part of the market-clearing process a passive *trading crowd*—a group of competitive potential market makers/dealers with order processing costs of  $r$  per share—provides unlimited liquidity by selling whenever  $p > v + r$  in either market. We denote the lowest price above  $v + r$  (the crowd's reservation asking price) as  $p_{\max}$ . This is an upper bound on the market-clearing price in each exchange.

Our crowd represents both professional dealers at banks and brokerage firms who regularly monitor trading in pure (electronic) limit order markets as well as the actual trading crowd physically on the floor of hybrid markets like the NYSE. In the pure market, we assume operationally that any excess demand  $B^p - \sum_{p_j \leq p_{\max}} S_j^p > 0$  that the PLM book cannot absorb is posted as a limit *buy* at  $p_{\max}$ , where the crowd then sees it and enters to take the other side of the trade. In the hybrid market, the specialist is first obligated to announce his cleanup price  $p^h$  and to give the crowd a chance to trade ahead of him before clearing the market. Hence the specialist cannot ask more than  $p_{\max-1}$  (i.e., one tick below  $p_{\max}$ ) and still undercut the trading crowd on large trades.

### 1.4 The specialist's order execution problem

The specialist has two advantages over other liquidity providers. First, he has a timing advantage over the value traders. He provides liquidity *ex post* (after seeing the realized size of the order  $B^h$ ), whereas limit orders, on both markets, are costly *ex ante* precommitments of liquidity. Second, he has a cost advantage over the trading crowd. Although we have singled out one specific trader and labeled him the "specialist," one could also view the market makers/dealers in the crowd as having *heterogeneous* order processing costs. All but one have costs  $r > 0$ , but one market maker/dealer has a competitive advantage in that his order processing/inventory costs are zero. Our



specialist is simply whichever dealer currently happens to be the lowest-cost liquidity provider in the market.

The specialist maximizes his profit from clearing the hybrid market by choosing a cleanup price  $p^h$  which, given the market order  $B^h$  and the HM book  $S_1^h, S_2^h, \dots$ , solves

$$\max_{v < p \leq p_{\max-1}} \pi(p | B^h) = \left[ B^h - \sum_{p_i \leq p} S_i^h \right] (p - v). \quad (2)$$

In particular, he sells at  $p^h$  after first executing all HM limit orders with priority.<sup>7</sup> The trade-off the specialist faces is that the higher the cleanup price, the more limit orders have priority, and thus, the fewer shares he personally sells at that price. The upper bound of  $p_{\max-1}$  is because the specialist must also undercut the HM crowd to trade.

In executing an arriving market order  $B^h$ , the specialist competes directly with the HM limit order book. Since he cannot profitably undercut limit orders at  $p_1$  (i.e., the lowest price above his valuation  $v$ ), he simply crosses small market orders,  $B^h \leq S_1^h$ , against the book and sets  $p^h = p_1$ . For larger orders,  $B^h > S_1^h$ , the specialist sets the cleanup price  $p^h$  so that he always sells a positive amount.<sup>8</sup> This implies that hybrid limit orders  $S_j^h$  at prices  $p_j > p_1$  either execute *in toto* or not at all. In contrast, there is only partial execution of any limit sells  $S_i^h$  at  $p_1$  when  $B^h < S_1^h$ .

From Seppi (1997) Proposition 1 we know that the specialist's optimal pricing strategy  $p^h(B^h)$  is monotone in the size of the arriving order  $B^h$ . Thus it can be described by a sequence of *execution thresholds* for order sizes that trigger execution at successively higher prices

$$\theta_j^h = \max \{ B^h \mid p^h(B^h) < p_j \}. \quad (3)$$

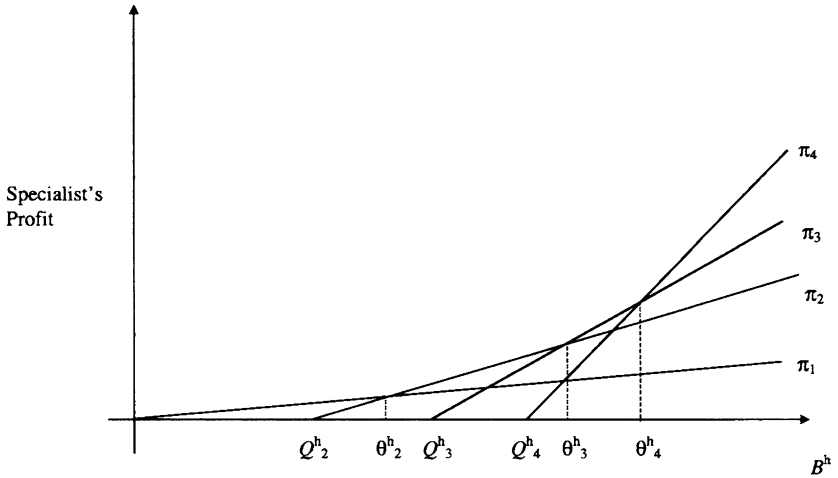
The cleanup price is  $p^h \geq p_j$  only when the arriving market order is sufficiently large in that  $B^h > \theta_j^h$ . Figure 2 illustrates this by plotting the specialist's profit from selling at different hypothetical prices  $p_j = p_1, \dots, p_{\max-1}$ ,

$$\pi_j = (B^h - Q_j^h)(p_j - v), \quad (4)$$

conditional on different possible orders  $B^h \geq Q_j^h$ . Lemma 3 in Section 1.8 shows that, in equilibrium, the execution thresholds  $\theta_j^h$  are determined, as shown here, by the adjacent prices  $p_{j-1}$  and  $p_j$ . When  $B^h$  is less than  $\theta_j^h$ , the profit  $\pi_{j-1}$  from selling at  $p_{j-1}$  is greater than  $\pi_j$ , while for  $B^h > \theta_j^h$  the profit  $\pi_j$  is

<sup>7</sup> The specialist only trades once. Selling additional shares at prices below  $p^h$  simply reduces the size of his (more profitable) cleanup trade at  $p^h$ . No submission costs  $c_j$  are incurred on the specialist's cleanup trade since ex post liquidity cannot be picked off.

<sup>8</sup> If  $p^h = p_1$  and  $B^h > S_1^h$ , then, by definition, the specialist is selling. If the specialist is not selling when  $p^h > p_1$ , then he went "too far" into the book. Lowering  $p^h$  would undercut some limit orders and thereby let the specialist sell some himself at a profit.



**Figure 2**  
**Specialist profit maximization and hypothetical HM limit order depths and execution thresholds**

$Q_j^h$  = cumulative depths in the HM limit order book,  $\pi_j$  = specialist's profit from selling at price  $p_j$  given a market order  $B^h > Q_j^h$ , and  $\theta_j^h$  = execution threshold for price  $p_j$ . This illustration assumes that  $Q_4^h > Q_3^h > Q_2^h > 0$ , where  $p_2 = p_{\min}$ . To be consistent with Lemma 3 below, the thresholds are *strictly* ordered so that  $\theta_j^h < \theta_{j+1}^h$  at all prices  $p_j$  with positive depth  $S_j^h > 0$ .

greater than  $\pi_{j-1}$ . When  $B^h = \theta_j^h$ , the specialist is indifferent between selling at  $p_{j-1}$  or  $p_j$ . To ensure that the active trader's Problem (1) is well defined and has a solution, we assume that the specialist uses the *lower* of these two prices and sets  $p^h(\theta_j^h) = p_{j-1}$ .<sup>9</sup> We summarize these properties in two ways:

- The largest market order that the active trader can submit such that the specialist will undercut the HM book at  $p_j$  by cleaning up at  $p_{j-1}$  is  $B^h = \theta_j^h$ . Orders larger than  $\theta_j^h$  are cleaned up at  $p_j$  or higher.
- For the value traders, their limit sells at  $p_j > p_1$  execute only if  $B^h > \theta_j^h$ .

Implicit in the specialist's maximization problem is the assumption that the specialist takes the arriving order  $B^h$  as given. In particular, he cannot influence the active trader's split between  $B^h$  and  $B^p$  by precommitting to sell at prices which undercut the rival PLM market, but which are *ex post time inconsistent* [i.e., do not satisfy Equation (2)]. This is equivalent to assuming that the specialist only sees the arriving hybrid order  $B^h$  (i.e., he cannot condition on the actual PLM order  $B^p$ ) and that he has no cost advantage in submitting limit orders of his own. With these assumptions, the only role for the specialist is *ex post* (or supplementary) liquidity provision as in Equation (2).

<sup>9</sup> If  $p^h(\theta_j^h) = p_j$  rather than  $p_{j-1}$ , then solving Problem (1) could involve trying to submit the *largest* order such that  $B^h < \theta_j^h$  in order to keep the HM cleanup price at  $p_{j-1}$ . Since this involves maximizing on an open set, no solution exists. This assumption also justifies the "max" rather than a "sup" in Equation (3).

## 1.5 Value traders

We model value traders as a continuum of individually negligible, risk-neutral Bertrand competitors. They arrive randomly at time 1, submit limit orders if profitable, and then leave.

The depths  $S_j^h$  and  $S_j^p$  at any price  $p_j$  in the two markets' respective limit order books are determined by the profitability of the marginal limit orders. Each market's book is open and publicly observable so that the expected profit on additional limit orders can be readily calculated. In the HM book the marginal expected profit on limit orders, given the specialist's execution thresholds, is

$$\begin{aligned} e_1^h &= \alpha \Pr(B^h \geq S_1^h)(p_1 - v) - c_1 \text{ at } p_1 \text{ and} \\ e_j^h &= \alpha \Pr(B^h > \theta_j^h)(p_j - v) - c_j \text{ at } p_j \end{aligned} \quad (5)$$

where  $\alpha \Pr(B^h \geq S_1^h)$  and  $\alpha \Pr(B^h > \theta_j^h)$  are the endogenous probabilities, given the distribution  $F^h$ , of a market order large enough to trigger execution of all HM limit sells at prices  $p_1$  or  $p_j$ , respectively.

In the PLM book, the cumulative depths  $Q_j^p$  play a role analogous to the specialist's execution thresholds in Equation (5). The marginal PLM limit sell at  $p_j$  is filled only if  $B^p$  is large enough to reach that far into the book. Thus the marginal expected profit at  $p_j$  is<sup>10</sup>

$$e_j^p = \alpha \Pr(B^p \geq Q_j^p)(p_j - v) - c_j. \quad (6)$$

Value traders do not *need* to trade per se. They simply submit limit orders until any expected profits in the PLM and HM limit order books are driven away. Since limit order submission is costly, limit orders are only posted at prices where there is a sufficiently high probability of profitable execution. To derive a lower bound on the set of possible limit sells, we note that the maximum expected profit at  $p_j$  is  $\alpha(p_j - v) - c_j$ . This is the expected profit if the limit order is always executed given any  $x > 0$ . From this it follows that limit orders at prices where  $p_j < v + \frac{c_j}{\alpha}$  are not profitable ex ante and hence are never used. We denote the lowest price such that limit sells are potentially profitable as  $p_{\min} = \min\{p_j \in \mathcal{P} \mid v + \frac{c_j}{\alpha} < p_j\}$  and note that  $p_j > v + \frac{c_j}{\alpha}$  for all prices  $p_j > p_{\min}$ . Natural upper bounds are  $p_{\max}$  (in the pure limit order market) and  $p_{\max-1}$  (in the hybrid market) since the PLM crowd and HM specialist undercut any limit sells above these prices. We make the following simplifying assumption about the relative ordering of  $p_{\min}$  versus  $p_1$  and  $p_{\max}$  in our analysis hereafter.

<sup>10</sup> Unlike in the hybrid market, partial execution of limit sells above  $p_1$  is possible in the PLM. However, the resulting ex ante profitability of inframarginal PLM limit orders does not affect the profitability of the marginal PLM limit orders and hence does not affect the equilibrium PLM depths  $S_j^p$ .

**Assumption 1.**  $p_1 < p_{\min} \leq p_{\max}$ .

The assumption  $p_{\min} \leq p_{\max}$  means that positive depth in one or both of the limit order books is possible. Otherwise the books would be empty. The assumption  $p_{\min} > p_1$  is the relevant case given the trend toward decimalization and finer price grids.<sup>11</sup> An immediate implication of  $p_1 < p_{\min}$  is that now the specialist always trades when a market order  $B^h > 0$  arrives. When the arriving buy order is small, he always has the option of selling one “tick” below the limit order book at  $p_{\min-1}$ , thereby undercutting all of his rival liquidity providers in the hybrid market.

### 1.6 The active trader

Our model differs from Seppi (1997) in that the active trader’s orders solve an optimization problem. In particular, recall that the active trader chooses her market orders  $B^p \geq 0$  and  $B^h \geq 0$  to minimize the total liquidity premium she pays to buy  $x$  shares,

$$\tau(x) = \min_{B^h, B^p \text{ s.t. } B^h + B^p = x} T^h(B^h) + T^p(B^p). \quad (7)$$

Given the actions of the liquidity providers described above, we now have explicit expressions for  $T^h$  and  $T^p$ . In the hybrid market—given the HM limit order book,  $S_{\min}^h, \dots$ , the associated thresholds  $\theta_2^h, \dots$ , and the crowd’s reservation price  $p_{\max}$ —the active trader faces a cost schedule

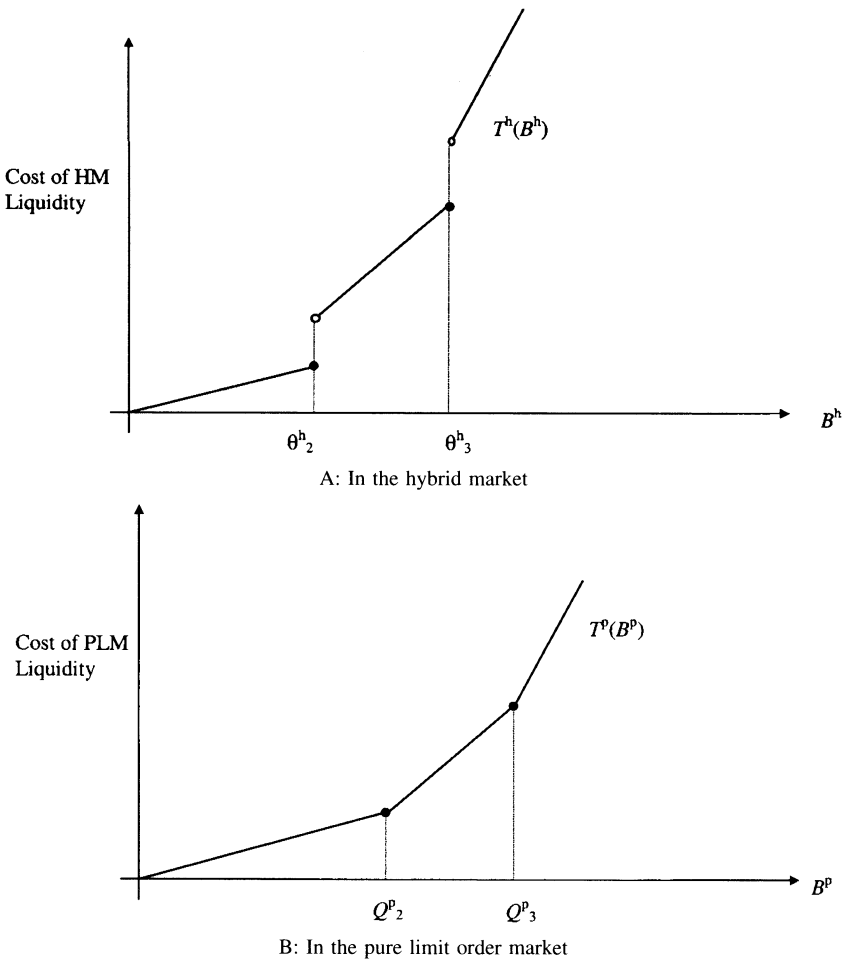
$$T^h(B^h) = \sum_{p_j \leq p^h} S_j^h p_j + (B^h - Q^h) p^h - B^h v, \quad (8)$$

where the first term is the cost of buying from the HM book and the second is the cost of any shares bought from the specialist at his cleanup price,  $p^h(B^h)$ . Recall that subtracting  $B^h v$  simply expresses trading costs as a *price impact* or liquidity premium in excess of the baseline valuation. In the pure limit order market—given the PLM book  $S_{\min}^p, \dots$  and the crowd’s  $p_{\max}$ —the active trader faces a liquidity cost schedule

$$T^p(B^p) = \sum_{p_i < p^p} S_i^p p_i + \left[ B^p - \sum_{p_i < p^p} S_i^p \right] p^p - B^p v. \quad (9)$$

As illustrated in Figure 3, the two schedules differ significantly. In the hybrid market,  $T^h$  has discontinuities at each of the specialist’s thresholds  $\theta_2^h, \dots, \theta_{\max-1}^h$ , whereas  $T^p$  in the pure market is continuous. The discontinuities in  $T^h$  arise because the specialist only provides liquidity at his cleanup

<sup>11</sup> This assumption unclutters the statement of our results by eliminating a number of special cases at  $p_1$  when  $p_{\min} = p_1$ . Since the specialist cannot profitably undercut the HM book at  $p_1$  when  $S_1^h > 0$ , limit sells at  $p_1$  are different from limit sells at prices  $p_j > p_1$ . Details about the  $p_{\min} = p_1$  case are available from the authors.



**Figure 3**  
Hypothetical liquidity cost schedules

price,  $p^h$ . If  $B^h = \theta_j^h$ , then the specialist crosses  $Q_{j-1}^h$  shares of the market order with limit orders at prices  $p_{\min}, \dots, p_{j-1}$  in the HM book and then sells the rest,  $\theta_j^h - Q_{j-1}^h$ , himself at  $p_{j-1}$ . However, once  $B^h$  exceeds  $\theta_j^h$  by even a small  $\epsilon > 0$ , the total cost of liquidity jumps, since undercutting the limit sells at  $p_j$  no longer maximizes the specialist's profit. Now, after crossing  $Q_{j-1}^h$  with the limit orders up through  $p_{j-1}$ , the remaining  $\theta_j^h + \epsilon - Q_{j-1}^h$  shares are executed at  $p_j$ . Of this,  $S_j^h$  comes from limit orders at  $p_j$  and  $\theta_j^h + \epsilon - Q_j^h$  is sold by the specialist. Thus when  $p^h$  reaches  $p_j$ , the specialist stops selling at  $p_{j-1}$ , and there is a discrete reduction in the liquidity available at the now inframarginal price  $p_{j-1}$ . In contrast, a higher stop-out price

$p^p$  in the pure limit order market has no effect whatsoever on inframarginal liquidity provision at lower prices in the PLM book.

An important fact about Equation (7) is that the active trader's problem may have multiple solutions for some  $x$ 's. This happens when the cost-minimizing cleanup/stop-out prices are equal,  $p^h = p^p = p_j$ , and there is some "slack" in the two liquidity supply schedules,  $x < \theta_{j+1}^h + Q_j^p$ . In this case, small changes in  $B^h$  and  $B^p$  do not change  $p^h$  and  $p^p$  or the overall total cost  $\tau(x)$  and, as a result, the active trader is indifferent about where she buys at  $p_j$ . For total volumes  $x$ , where the solution to Equation (7) is unique, the active trader's orders,  $B^h$  and  $B^p$ , are entirely determined by cost minimization. However, when multiple solutions exist, her choice of which particular cost-minimizing pair of orders,  $B^h$  and  $B^p$ , to submit depends on a "tie-breaking" order preferencing rule.

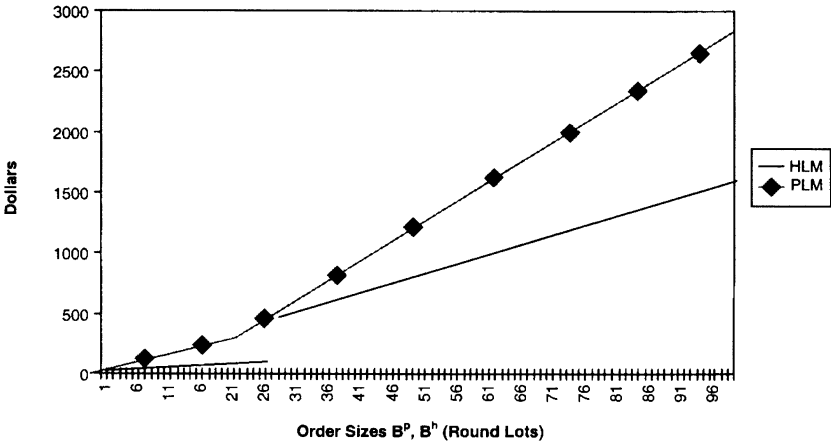
**Definition 1.** An order preferencing rule,  $\kappa$ , is a family of probability distributions  $\{\kappa_x\}$  over the set of cost-minimizing orders  $B^h$  and  $B^p = x - B^h$  solving Equation (7) indexed by the total volumes  $x$ .

The order preferencing rule  $\kappa$ , together with the distribution  $F$  over  $x$ , induce, in turn, endogenous distributions  $F^p$  and  $F^h$  over the orders  $B^p$  and  $B^h$  arriving in the two markets. In practice, investors may preference one market over another out of habit or because of "payment for order flow" or locational convenience. While our notation allows for preferencing to be deterministic, randomized, and/or contingent on the total volume  $x$ , this article focuses on two polar cases in which either the pure or the hybrid market is consistently preferred.

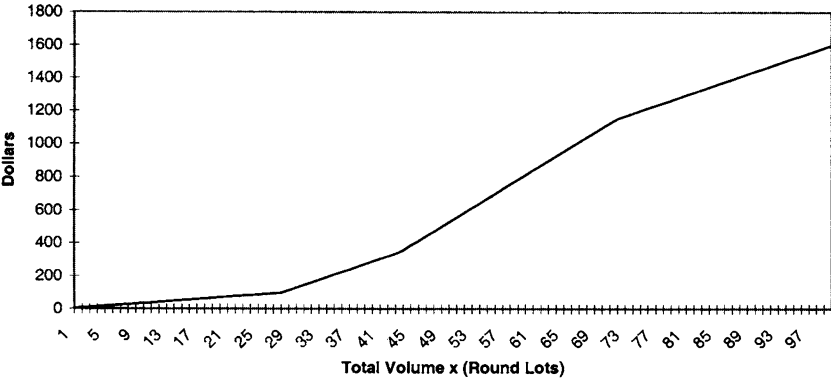
### 1.7 Numerical example

Figure 4 illustrates the choice of the active trader's order submission strategy  $B^p(x)$  and  $B^h(x)$ . In this example the active investor minimizes her total trading costs given the two liquidity supply schedules  $T^p$  and  $T^h$  in Figure 4a, where  $p_1 = \$30.125$ ,  $p_2 = \$30.25$ , and  $p_3 = p_{\max} = \$30.375$  and where the stock's common valuation is  $v = \$30.09$  per share. We show in Section 2 that these particular schedules can be supported in equilibrium given a specific order preferencing rule.

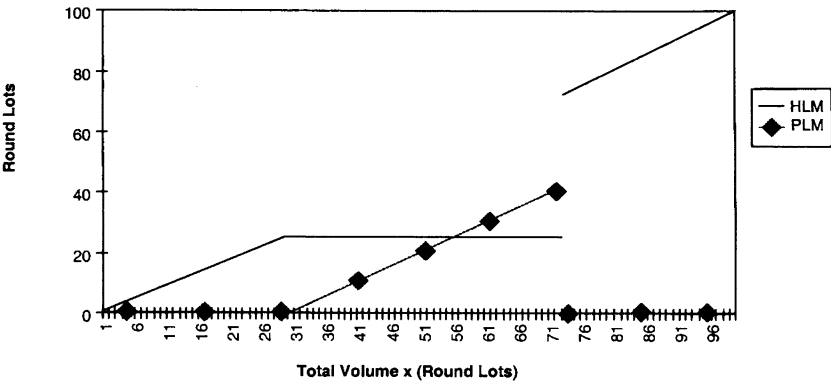
Figure 4b depicts the *minimized* aggregate cost schedule  $\tau$  corresponding to  $T^h$  and  $T^p$  and Figure 4c shows a pair of cost-minimizing order submission strategies  $B^h(x)$  and  $B^p(x)$ . If the active trader needs to buy  $x \leq 28.1$  round lots, then her costs are minimized by buying  $B^h = x$  in the hybrid market at a marginal cost of liquidity of  $p_1 - v = 0.035$  cents per share. When  $28.1 < x < 71.9$  she optimally caps her order to the hybrid market at  $B^h = 28.1$  (i.e., avoiding the discontinuity above 28.1) and buys  $B^p = x - 28.1$  round lots in the pure market at a marginal cost of  $p_2 - v = 0.16$  cents per share (if  $B^p \leq 15.6$ ) or  $p_3 - v = 0.285$  cents thereafter (if  $15.6 < B^p < 43.8$ ).



A: HM and PLM liquidity supply schedules



B: Minimized aggregate liquidity supply schedule



C: Optimal market order submissions

**Figure 4**  
**Example of optimal market order submission strategies and liquidity cost schedules**  
The numerical parameter values are the same as in Figure 8.

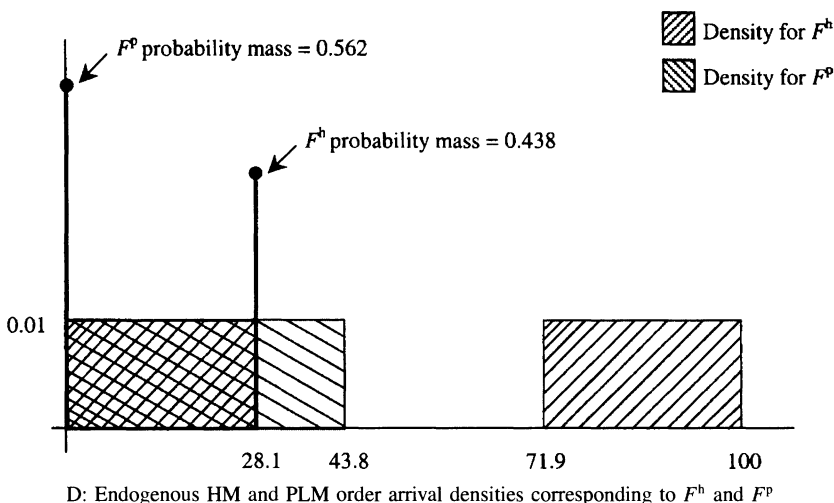


Figure 4  
(continued)

When  $x \geq 71.9$  her costs are minimized by any combination of orders  $B^p \leq 15.6$  and  $B^h = x - B^p$ , since she is indifferent about where to buy the last 15.6 round lots (i.e., since the price is  $p_2$  in either market). An order preferencing rule is needed to pin down  $B^h$  and  $B^p$  in this region. In Figure 4c we assume that, when indifferent, the active investor favors the hybrid market. We consider this and other alternative preferencing rules in greater detail below.

Figure 4d shows the densities corresponding to the order arrival distributions  $F^h$  and  $F^p$  induced by  $B^h(x)$  and  $B^p(x)$  given an additional assumption that the total volume  $x$  is distributed uniformly over  $[0, 100]$  round lots. As this example illustrates, there can be endogenous “flat regions” (i.e., densities equal to zero) and probability mass points in  $F^h$  and  $F^p$  even when the total volume distribution  $F$  is continuous and increasing. In particular, the active trader’s efforts to avoid the jump in the hybrid market liquidity supply schedule  $T^h$  leads to the mass point  $\Pr(28.1 < x < 71.9) = 0.438$  at  $B^h = 28.1$  in  $F^h$  and hybrid preferencing leads to the mass point  $\Pr(x \leq 28.1) + \Pr(x \geq 71.9) = 0.562$  at  $B^p = 0$  in  $F^p$ . The location of such mass points will play an important role in the equilibrium interdependencies linking liquidity supply (i.e., limit orders and the specialist’s cleanup decision) and liquidity demand (i.e., the market order split).

## 1.8 Equilibrium

Given the market participants and their actions, an equilibrium is defined as follows.



**Definition 2.** A Nash equilibrium is a set of depths, thresholds, and order arrival distributions  $\{S_1^p, S_2^p, \dots, S_1^h, S_2^h, \dots, \theta_2^h, \dots, F^h, F^p\}$  such that

- The value traders' marginal expected profit at each price  $p_j$  in the pure limit order book is nonpositive,  $e_j^p \leq 0$ , if  $S_j^p = 0$  and zero,  $e_j^p = 0$ , if  $S_j^p > 0$ .
- Similarly, in the hybrid market  $e_j^h \leq 0$  if  $S_j^h = 0$  and  $e_j^h = 0$  if  $S_j^h > 0$ .
- The specialist's execution thresholds  $\theta_2^h, \dots$  satisfy his profit maximization condition [Equation (3)].
- The market order arrival distributions  $F^h$  and  $F^p$  are consistent with the active trader's cost minimization problem [Equation (7)], the volume distribution  $F$ , and a preferencing rule  $\kappa$ .

The endogeneity of  $F^h$  and  $F^p$  is critical to the definition and construction of equilibrium in our model. As seen in Figure 4, the order arrival distributions can be complicated with probability mass points and flat regions. If  $F^h$  and  $F^p$  were *exogenous*, then competition might not drive expected profits to zero since the limit order execution probabilities are not continuous around exogenously fixed mass points. Indeed, Seppi (1997) shows that, with fixed mass points in exogenous distributions  $F^h$  and  $F^p$ , the proper competitive conditions (when depth is positive) are weak inequalities  $e_j^h \geq 0$  and  $e_j^p \geq 0$ —rather than strict break-even conditions  $e_j^h = 0$  and  $e_j^p = 0$  as above.

In our model, however, the location of any mass points is endogenously determined by the active trader's cost minimization problem [Equation (7)] and her preferencing rule  $\kappa$ . Notice that, collectively, the value traders are competitive first movers. Given the liquidity supply schedules  $T^h$  and  $T^p$ , the active trader chooses  $B^h$  and  $B^p$  to minimize her trading costs. The schedule  $T^p$  is, in turn, determined by the submitted limit order book  $S_{\min}^p, \dots$  in the pure limit order market. The hybrid schedule  $T^h$  is determined by the HM book  $S_{\min}^h, \dots$  both directly and also indirectly through the specialist's thresholds  $\theta_j^h$ . Thus the execution probability  $\Pr(B^h > \theta_j^h)$  in the definition of  $e_j^h$  in Equation (5) depends on the depth  $S_j^h$  both mechanically—in that changing  $S_j^h$  changes  $\theta_j^h$ , holding the distribution  $F^h$  fixed—and, more fundamentally, in that changing  $S_j^h$  changes the distribution of arriving market orders  $F^h$  itself via the impact of  $S_j^h$  on  $T^h$  and hence on the split between  $B^h$  and  $B^p$ . An analogous argument holds in the pure limit order market. In light of the endogeneity of  $F^h$  and  $F^p$ , the execution probabilities in Equations (5) and (6) can be written more explicitly as

$$\begin{aligned} \Pr(B^h > \theta_j^h) &= \Pr(B^h > \theta_j^h(S_1^h, \dots) \mid F^h[T^h(S_1^h, \dots), T^p(S_1^p, \dots), \kappa]) \\ \Pr(B^p \geq Q_k^p) &= \Pr(B^p \geq Q_k^p \mid F^p[T^h(S_1^h, \dots), T^p(S_1^p, \dots), \kappa]). \end{aligned} \quad (10)$$

**Lemma 1.** The limit order execution probabilities  $\Pr(B^h > \theta_j^h)$  and  $\Pr(B^p \geq Q_j^p)$ , and hence the marginal expected profits  $e_j^h$  and  $e_j^p$  are continuous functions of  $S_j^h$  and  $S_j^p$ , respectively.

With continuous expected profits  $e_j^h$  and  $e_j^p$ , the competitive profit-seeking behavior of the value traders ensures that expected profits are driven to zero in equilibrium.<sup>12</sup>

A number of insights follow directly from the break-even property of the equilibrium limit order books. One of the most useful is that rewriting the break-even condition  $e_j^p = 0$  using Equation (6) gives the equilibrium probabilities of execution for the marginal PLM limit orders:

$$\Pr(B^p \geq Q_j^p) = \frac{c_j/\alpha}{p_j - v} \quad \text{at each } p_j \geq p_{\min}, \text{ where } S_j^p > 0. \quad (11)$$

Similarly  $e_j^h = 0$  and Equation (5) give the equilibrium HM execution probabilities:

$$\Pr(B^h > \theta_j^h) = \frac{c_j/\alpha}{p_j - v} \quad \text{at each } p_j \geq p_{\min}, \text{ where } S_j^h > 0. \quad (12)$$

The strong versus the weak inequalities in Equations (11) and (12) simply reflect the difference in limit order execution in the two markets.

The key step when computing equilibria in Section 2 is to represent the endogenous distributions  $F^h$  and  $F^p$ —from which the probabilities  $\Pr(B^h > \theta_j^h)$  and  $\Pr(B^p \geq Q_j^p)$  are computed—in terms of the exogenous total volume distribution  $F$ . Substituting these representations into Equations (11) and (12) lets us solve for the equilibrium limit order books in the two markets. One additional piece of notation will be useful when doing this. Let  $H$  denote the inverse of the total volume distribution  $F$ , where  $\Pr(x > H(z)) = z$ , and define

$$H_j = \begin{cases} H\left(\frac{c_j/\alpha}{p_j - v}\right) & \text{for } p_j = p_{\min}, \dots, p_{\max} \text{ so that } \frac{c_j/\alpha}{p_j - v} \leq 1 \\ 0 & \text{otherwise.} \end{cases} \quad (13)$$

Since  $F$  is continuous and strictly increasing, the existence and uniqueness of the  $H_j$ 's is guaranteed.

Another implication of the limit order break-even property is that since the specialist faces the same ex ante costs  $c_j$  of having limit orders picked off as do the value traders, and since the value traders compete away any expected limit order profits, the specialist does not submit limit orders of his own. In addition, the break-even property means that the HM book and execution thresholds have the same simple structure in our model as in Seppi's (1997) Proposition 2. We restate this result here in two parts.

<sup>12</sup> Our assumption that the value traders are individually negligible—that is, that they individually take the aggregate depths  $S_j^h$  and  $S_j^p$ , and hence the distributions  $F^h$  and  $F^p$  as given—simplifies the definition of equilibrium since it means we only need to check that the limit order books break even locally. In particular, the profitability of noninfinitesimal deviations that change the depths  $S_j^h$  and  $S_j^p$  does not need to be checked.

**Lemma 2.** *In the hybrid market, if  $S_j^h > 0$  and  $p_{j+1} < p_{\max}$ , then  $S_{j+1}^h > 0$ .*

**Lemma 3.** *In equilibrium the specialist's execution thresholds are strictly ordered  $\theta_j^h < \theta_{j+1}^h$  at prices with positive depth  $S_j^h > 0$  such that*

$$\theta_j^h = Q_{j-1}^h + \frac{1}{\gamma_j} S_j^h \quad (14)$$

where

$$\gamma_j = \frac{p_j - p_{j-1}}{p_j - v}. \quad (15)$$

Lemma 2 says that there are no “holes” in the HM book (e.g., positive depths at  $p_{j-1}$  and  $p_{j+1}$ , but  $S_j^h = 0$ ). Lemma 3 justifies the claim in Figure 2 that the thresholds are determined by comparing the profits at adjacent prices. The term  $\frac{1}{\gamma_j} > 1$  in Equation (14) measures how aggressively the specialist undercuts the hybrid limit order book at  $p_j$  by selling at  $p_{j-1}$ .

## 2. Results About Competition

Jointly modeling the supply and demand of liquidity lets us investigate the equilibrium impact of intermarket competition on both limit order placement and the market order flow. As barriers to trade fall (e.g., with improved telecommunications), a natural “feedback” loop seems to push toward a concentration of liquidity and trading. A market which attracts more market orders will tend to attract more limit orders which, in turn, makes that market more liquid and thus even more attractive to market orders.<sup>13</sup> On its face, this might suggest that a single centralized market is the inevitable end state for the financial marketplace.

Glosten (1994) predicts further that trading and liquidity will concentrate in a single virtual competition-proof limit order market. The analysis leading to this prediction assumes, however, that the liquidity providers all face identical costs and that the timing of their liquidity provision decisions is the same (i.e., everyone must quote *ex ante* to participate). If, however, costs are heterogeneous and if liquidity is both *ex post* and *ex ante*, is a centralized marketplace still inevitable? Or is the coexistence of competing exchanges

<sup>13</sup> Admati and Pfleiderer (1988) and Pagano (1989) were the first to study the concentration of order flow and its connection with market liquidity.

possible? In answering these questions we focus particularly on the viability of different markets' respective limit order books.

**Definition 3.** *The book in market I (either the HM or PLM) dominates the book in the other market II if limit orders  $S_j^I > 0$  are posted at at least one price  $p_j$  in market I and if the limit order book in market II is empty,  $S_k^{II} = 0$ , at each price  $p_k$ ,  $k = 1, \dots, j_{\max}$ .*

This criterion is weaker than competition-proofness in Glosten (1994) since the specialist (or crowd) may still trade even when the hybrid (pure) book is empty. If both books have positive depth,  $S_j^h > 0$  and  $S_k^p > 0$  at (possibly different) prices  $p_j$  and  $p_k$ , we say the two markets *coexist*.

## 2.1 General results

Each of the two exchanges has distinct advantages relative to the other market. On the one hand, the specialist has the lowest ex post cost of providing liquidity. On the other, the continuity of the PLM liquidity supply schedule  $T^p$  makes the pure limit order market attractive for market orders.

**Lemma 4.** *If the hybrid cleanup price is  $p^h = p_j > p_{\min}$ , then all PLM limit sells at least up through  $p_{j-1}$  are executed in full,  $B^p \geq Q_{j-1}^p$ , and thus  $p^p \geq p_{j-1}$ .*

**Lemma 5.** *The smallest total volume,  $\inf\{x \mid B^h(x) > \theta_j^h\}$ , such that any HM limit sells at  $p_j > p_{\min}$  are executed in full is strictly larger than the corresponding volume,  $\inf\{x \mid B^p(x) \geq Q_j^p\}$ , for any PLM limit sells at  $p_j$ .*

The reason for the asymmetry between the two markets is that the HM liquidity supply schedule  $T^h$  is discontinuous at the execution thresholds  $\theta_j^h$ , whereas the PLM liquidity supply schedule  $T^p$  is continuous. Thus if  $B^h(x) = \theta_j^h$  and  $B^p(x) = Q_{j-1}^p$  for a volume  $x = \theta_j^h + Q_{j-1}^p$ , then, given a slightly larger volume  $x + \epsilon$ , the active trader always buys the additional  $\epsilon$  shares in the pure limit order market. Using the PLM as a buffer or “pressure valve” in this way lets her keep  $B^h = \theta_j^h$  and thereby avoid the discontinuous jump in  $T^h$  above  $\theta_j^h$  (as in Figure 3a). Indeed, since higher stop-out prices  $p^p$  increase only the *slope* of the PLM cost schedule  $T^p$  (see Figure 3b), she is even willing to buy a small number of shares at  $p_{j+1}$  and potentially at even *higher* prices in the PLM so as to keep the HM cleanup price at  $p_{j-1}$ . It is this “pressure valve” role of the pure market that leads to Lemma 5. Of course, once  $x$  is sufficiently large, it is cheaper to increase  $B^h$  above  $\theta_j^h$  rather than to keep buying ever larger quantities  $x - \theta_j^h - Q_j^p$  at progressively higher premia  $p^p - p_j$  indefinitely in the pure market. In doing so, the active investor naturally scales back her premium PLM buying at prices  $p_{j+1}, \dots$

Putting Lemmas 4 and 5 together does not imply that  $p^p \geq p^h$ . Cost minimization implies that, for some total volumes  $x$ , the marginal limit sell at  $p_j$  in the PLM book is *optimally* executed (i.e.,  $B^p(x) \geq Q_j^p$ ) while the marginal order in the HM book is unexecuted (i.e.,  $B^h(x) \leq \theta_j^h$ ). However, due to preferencing, the marginal HM limit sell at  $p_j$  may execute when the marginal PLM limit sell does not when the cost-minimizing split,  $B^h$  and  $B^p$ , is not unique. Using this last observation, the probability of execution for the marginal limit sell in the HM book can be written as

$$\begin{aligned} \Pr(S_j^h \text{ executes}) &= \Pr(\text{both } S_j^h \text{ and } S_j^p \text{ execute}) \\ &\quad + \Pr(S_j^h \text{ executes, but not } S_j^p \text{ due to preferencing}) \end{aligned} \quad (16)$$

while the corresponding probability in the PLM book is

$$\begin{aligned} \Pr(S_j^p \text{ executes}) &= \Pr(\text{both } S_j^h \text{ and } S_j^p \text{ execute}) \\ &\quad + \Pr(S_j^p \text{ executes, but not } S_j^h \text{ due to preferencing}) \\ &\quad + \Pr(S_j^p \text{ executes, but not } S_j^h \text{ due to cost} \\ &\quad \text{minimization}). \end{aligned} \quad (17)$$

Thus the only way to support an equilibrium in which limit orders at  $p_j$  in the HM book coexist with (or dominate) limit sells in the PLM book—that is, in which  $e_j^h = 0 \geq e_j^p$ —is if the active trader's preferencing rule  $\kappa$  favors the hybrid market frequently enough in that

$$\begin{aligned} &\Pr(S_j^h \text{ executes, but not } S_j^p \text{ due to preferencing}) \\ &\geq \Pr(S_j^p \text{ executes, but not } S_j^h \text{ due to preferencing}) \\ &\quad + \Pr(S_j^p \text{ executes, but not } S_j^h \text{ due to cost minimization}). \end{aligned} \quad (18)$$

Hence the order preferencing rule  $\kappa$  plays a central role in supporting any equilibrium with limit orders in the hybrid book.

## 2.2 Pure market order preferencing

An immediate implication of Inequality (18) is that if the active trader, when indifferent, *always* preferences the pure limit order market over the hybrid market, then the HM book is empty.

**Definition 4.** *With pure market preferencing the active trader, when indifferent, always sends the largest order  $B^p$  to the pure limit order market such that  $(x - B^p, B^p)$  solves Equation (7) for  $x$ .*

**Proposition 1.** *Given pure market preferencing, an equilibrium exists and has a dominant PLM book (DPLM) where*

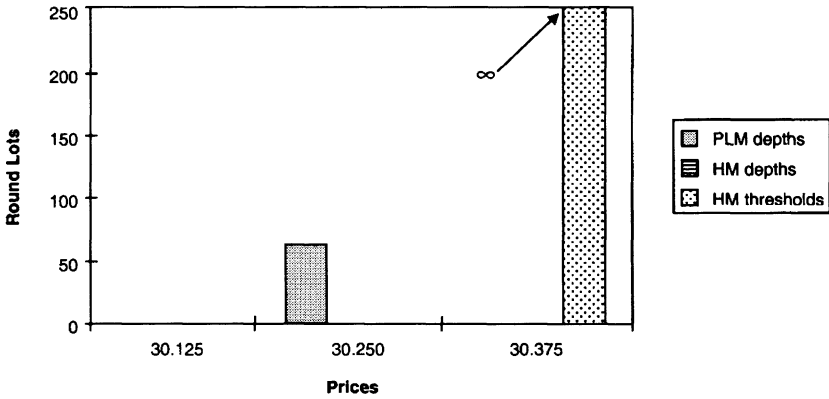
- The pure limit order book has positive depths at prices  $p_{\min}, \dots, p_{\max-1}$  given by

$$S_j^{DPLM} = H_j - H_{j-1}, \quad (19)$$

- The hybrid limit order book is empty,  $S_j^h = 0$ , at all  $p_j$ , and
- The active trader optimally splits her order, sending  $B^p = \min\{x, H_{\max-1}\}$  to the PLM and buying any residual,  $B^h = \max\{0, x - H_{\max-1}\}$ , from the specialist in the hybrid market.

Given an empty HM book, the active trader's orders are directed first to the pure limit order market,  $B^p = x$ , until the available liquidity up through  $p_{\max-1}$  is exhausted. This lets us identify the endogenous probability of execution  $\Pr(B^p \geq Q_j^p)$  with PLM preferencing as  $\Pr(x \geq Q_j^p)$ . Substituting this in Equation (11) and then recursively inverting using the identity  $Q_j^p = Q_{j-1}^p + S_j^p$  gives the equilibrium PLM book in Equation (19). Once the active trader's total volume is too large for the PLM limit order book alone,  $x > Q_{\max-1}^p = H_{\max-1}$ , she caps  $B^p$  at  $H_{\max-1}$  and sends the rest,  $B^h = x - H_{\max-1}$ , to the hybrid market where the specialist (given the empty HM book) just undercuts the crowds'  $p_{\max}$  by selling at  $p_{\max-1}$ . Thus the HM order arrival distribution  $F^h$  has an endogenous mass point at  $B^h = 0$  equal to  $\Pr(x \leq H_{\max-1})$  and  $F^p$  has a mass point at  $B^p = H_{\max-1}$  equal to  $\Pr(x \geq H_{\max-1})$ . This is the unique equilibrium with pure market preferencing since, with a strictly increasing  $F$ , the  $H_j$ 's are unique. Figure 5 is a numerical example of this equilibrium.

The comparative statics for the equilibrium are intuitive. If the demand for sell liquidity increases—that is, if the probability  $\alpha$  of a market buy order

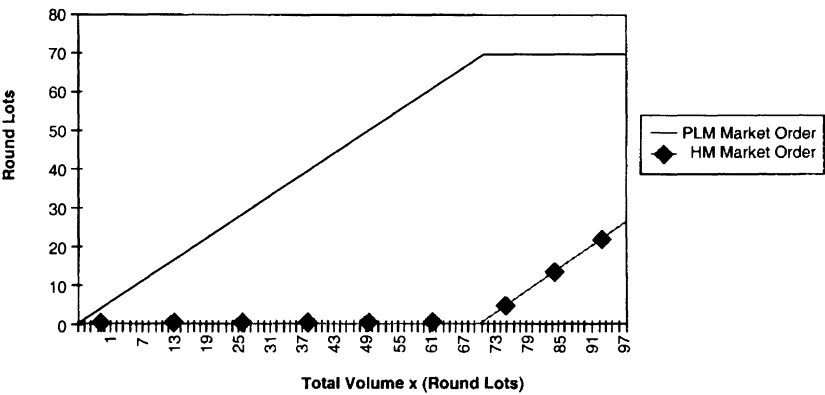


A: A dominant PLM equilibrium with PLM preferencing

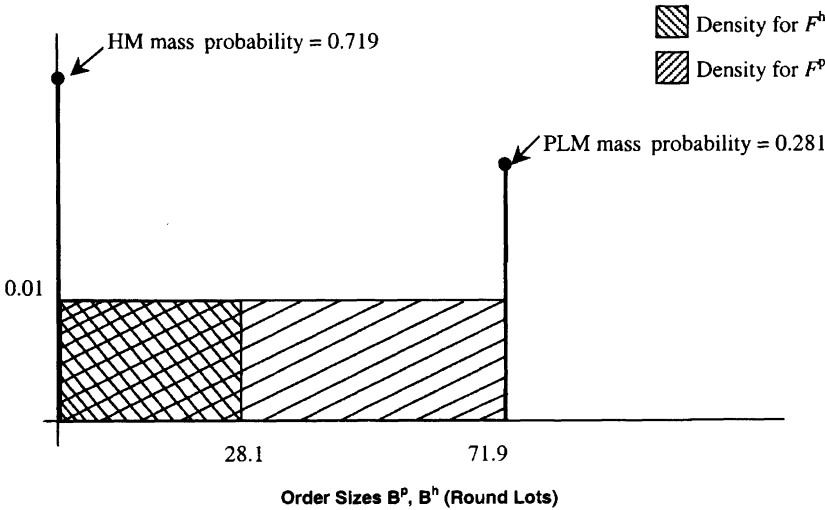
Figure 5

A dominant PLM equilibrium with PLM preferencing

Parameter values: common value  $v = \$30.09$ , ex ante limit order submission costs  $c_1 = \$0.0263$ ,  $c_2 = \$0.0225$ ,  $c_3 = \$0.0188$ , probability of a buy  $\alpha = 0.5$ ,  $p_{\max} = \$30.375$ , volume  $|x|$  uniform over  $[0, 100]$ .



B: Optimal market orders  $B^p, B^h$  in a DPLM equilibrium with PLM preferencing



C: Endogenous HM and PLM order arrival densities corresponding to  $F^h$  and  $F^p$

Figure 5  
(continued)

increases or if the total volume distribution  $F$  shifts to the right in the sense of first-degree stochastic dominance—then the cumulative depths  $Q_j^p = H_j$  are weakly larger. In other words, increased demand for liquidity induces, in equilibrium, greater liquidity supply. The same is true if the limit order submission costs  $c_j$  decrease.

It may be surprising that the specialist—despite his status as the “lowest cost” liquidity provider—is marginalized on all but the largest trades with PLM preferencing. This outcome is a consequence of his inability to offer credible price improvement when there are no HM limit orders to undercut. Once a market order  $B^h$  is in hand, the specialist has no incentive, given the empty HM book, to offer any price improvement below  $p_{\max-1}$ . We can

also show that alternative commitment mechanisms do not eliminate this equilibrium.

One obvious mechanism for committing to offer price improvement would seem to be for the specialist to post binding bid/ask quotes of his own. Recall, however, that the specialist has no cost advantage in precommitting to provide liquidity *ex ante*. Consequently any bid/ask quotes he might post incur the same costs  $c_j$  of being picked off as the value traders' limit orders.<sup>14</sup> Since the DPLM limit order book is break-even, any HM quotes from the specialist would trade at an expected loss *after* (given PLM preferencing) the PLM limit sells. Thus the specialist cannot profitably use binding quotes to attract market orders away from a competitive DPLM book.

If the specialist cannot precommit *ex ante* to provide price improvement, why can't the active trader use HM limit *buy* orders at price  $p_j < p_{\max-1}$  to force price improvement from him *ex post*? We can show, however, that this second mechanism also fails in the following sense:

**Proposition 2.** *The DPLM equilibrium can be supported as a Bayesian Nash equilibrium with beliefs that deter the use of limit buy orders above  $v$  and below  $p_{\max-1}$  by the active trader.*

The key step in the proof is to notice that when the specialist sees a limit order buy from the active trader, he does not know—given that he cannot see  $x$  directly or infer it by seeing  $B^p$  in the pure market—whether the active trader is trying to bluff him into offering extra price improvement relative to  $p_{\max-1}$ . Giving him sufficiently suspicious beliefs supports HM limit buys above  $v$  as off-equilibrium events.

On the surface, Proposition 1 is similar to Glosten's (1994) competition-proof result for pure limit order books. There are, however, two differences. First, our specialist does provide superior residual liquidity on the "back end,"  $x - H_{\max-1}$ , of large blocks by selling at  $p_{\max-1}$ . This is because our liquidity providers have heterogeneous costs—zero (*ex post*) for the specialist versus  $c_j$  (*ex ante*) for the value traders. Hence the PLM book is dominant but not competition-proof. Second, other preferencing rules support other equilibria. In particular, depending on how strongly the preferencing rule  $\kappa$  favors the hybrid market, it is possible, from Inequality (18), to support equilibria in which the HM and PLM books both coexist or even to have a dominant hybrid book.

### 2.3 Hybrid market order preferencing

Having seen that pure market preferencing leads to a dominant PLM equilibrium, it is natural to ask whether the polar opposite rule, hybrid market

<sup>14</sup> One example of dealers' vulnerability on this score is the evidence that lagging Nasdaq quotes are picked off on the small order execution system (SOES) by SOES bandits. See Harris and Schultz (1998) and Foucault, Roell, and Sandas (2003).



preferencing, is sufficient to offset the PLM's cost minimization advantage in Inequality (18).

**Definition 5.** *With hybrid market preferencing, the active trader, when indifferent, always sends the largest order  $B^h$  to the hybrid market such that  $(B^h, x - B^h)$  solves Equation (7) for  $x$ .*

This is clearly the maximum possible preferencing of the hybrid market, but even HM preferencing may be insufficient for HM dominance. In the rest of this section we identify market parameterizations for which HM preferencing leads to a dominant hybrid book when  $p_{\max} > p_2$ .<sup>15</sup> We also show that, outside of these parameter values, the two markets' books coexist with HM preferencing.

In a dominant HM equilibrium the PLM limit order book is, by definition, empty. Consequently the active trader's only alternative to the hybrid market is buying at  $p_{\max}$  from the PLM crowd. Thus we start by asking: How large must the total volume  $x$  be for the active trader to send an order  $B^h > \theta_j^h$  to the hybrid market if the PLM crowd is the only pressure valve?

**Lemma 6.** *With HM preferencing, the market order  $B^h$  submitted to the hybrid market is weakly increasing in the total target volume  $x$ .*

**Lemma 7.** *In a dominant HM equilibrium, the active trader buys  $B^h > \theta_j^h$  with a cleanup price  $p_j \geq p_{\min}$  in the hybrid market when*

$$\theta_{j+1}^h \geq x \geq \beta_j \equiv Q_{j-1}^h + S_j^h \frac{\psi_j}{\gamma_j} > \theta_j^h, \quad (20)$$

where  $\gamma_j < 1$  is given in Equation (15) and

$$\psi_j = \frac{p_{\max} - p_{j-1}}{p_{\max} - p_j} > 1. \quad (21)$$

In the appendix we show that the critical volume  $x = \beta_j$  is the solution to

$$\sum_{p_k \leq p_{j-1}} S_k^h p_k + (\theta_j^h - Q_{j-1}^h) p_{j-1} + (x - \theta_j^h) p_{\max} = \sum_{p_\ell \leq p_j} S_\ell^h p_\ell + (x - Q_j^h) p_j. \quad (22)$$

The left-hand side of this equation is the cost,  $T^h(\theta_j^h) + T^p(x - \theta_j^h)$ , of buying  $B^h = \theta_j^h$  in the hybrid market (so as to keep  $p^h = p_{j-1}$ ) and buying the rest at  $p_{\max}$  in the pure market. The right-hand side is the cost,  $T^h(x) + T^p(0)$ , of buying everything in the hybrid market and accepting the

<sup>15</sup> Given the restriction in Assumption 1, the only other possibility is  $p_{\max} = p_2 = p_{\min}$ , in which case both the HM and the PLM books are empty since the specialist provides unlimited liquidity at  $p_{\max-1} = p_1$ .

higher cleanup price  $p^h = p_j$ . When  $x = \beta_j$  the active investor is indifferent between these two alternatives so that, given HM preferencing,  $B^h(\beta_j) = \beta_j$  and  $B^p(\beta_j) = 0$ . The restriction  $\theta_{j+1}^h \geq \beta_j$  in Inequality (20) simply ensures that the specialist is, in fact, willing to clean up at  $p_j$ .

The term  $\psi_j$  is a measure of the relative cost of using the PLM crowd as a pressure valve to avoid raising  $B^h$  above  $\theta_j^h$ . Fix, for example, a particular price  $p_j$  and then let the cost of trading with the crowd increase without bound,  $p_{\max} \rightarrow \infty$ . We see that  $\psi_j \rightarrow 1$  and  $\beta_j \rightarrow \theta_j^h$ . Thus the active trader diverts less of her trading to the PLM as the cost of the pressure valve increases.

We use Lemmas 6 and 7 to express the *endogenous* execution probability  $\Pr(B^h > \theta_j^h)$  as the probability  $\Pr(x \geq \beta_j)$ . Substituting this into the break-even condition of Equation (12) and inverting gives

$$\beta_j = H_j, \quad (23)$$

which then, using Equation (20), can be recursively rearranged to get the HM limit order book when a dominant HM equilibrium exists.

**Lemma 8.** *In a dominant HM (DHM) equilibrium the break-even HM book is*

$$S_j^{DHM} = \begin{cases} 0 & \text{for } p_j < p_{\min} \\ (H_j - Q_{j-1}^{DHM}) \frac{\gamma_j}{\psi_j} & \text{for } p_j = p_{\min}, \dots, p_{\max-1}. \end{cases} \quad (24)$$

The term  $\frac{1}{\psi_j} < 1$  describes how the active trader's strategic use of the PLM crowd as a pressure valve thins out the HM limit order book. Similarly  $\gamma_j < 1$  describes the negative impact of strategic undercutting by the specialist on the HM book.

Lemma 8 characterizes the dominant HM book *provided* that a dominant HM equilibrium in fact exists. HM preferencing alone is not, however, sufficient to guarantee a dominant HM equilibrium. Existence hinges on two conditions.

**Condition 1.** The specialist must be *willing*, as noted in Inequality (20), to sell  $\beta_j - Q_j^h$  shares at  $p_j$  in that  $\beta_j \leq \theta_{j+1}^h$ .

**Condition 2.** Value traders do not want to post limit orders in the PLM book. This means that  $e_j^p \leq 0$  given the hybrid depths  $S_j^{DHM}$  from Lemma 8 and an otherwise empty PLM book.

Consider the specialist's willingness to trade first. Condition 1 is always satisfied at  $p_{\max-1}$  (since  $\beta_{\max-1} < \theta_{\max}^h = \infty$ ), so consider prices  $p_{\min}, \dots, p_{\max-2}$ .

Substituting Equation (24) into Equation (14) gives the dominant HM execution thresholds

$$\theta_{j+1}^{DHM} = M_{j+1} \equiv \frac{1}{\psi_{j+1}} H_{j+1} + \left(1 - \frac{1}{\psi_{j+1}}\right) \times \left[ \frac{\gamma_j}{\psi_j} H_j + \sum_{2 \leq i \leq j-1} \frac{\gamma_i}{\psi_i} \prod_{i+1 < k \leq j} \left(1 - \frac{\gamma_k}{\psi_k}\right) H_i \right] \quad (25)$$

in terms of the total volume distribution parameters  $H_2, \dots, H_j$ . Using this expression for  $\theta_{j+1}^h$  we can formalize Condition 1 as a restriction on the volume distribution  $F$ .

**Lemma 9.** *A necessary condition for a dominant HM equilibrium with HM preferencing is that at each price  $p_j = p_{\min}, \dots, p_{\max-2}$*

$$H_j \leq M_{j+1} \quad (26)$$

so that  $\beta_j \leq \theta_{j+1}^{DHM}$  and thus  $p^h(\beta_j) = p_j$ .

Next, consider Condition 2 for PLM profitability condition. For the pure limit book to be empty, the expected marginal profit on PLM limit sells must be  $e_j^p \leq 0$  when  $S_1^p = \dots = S_{\max}^p = 0$  given the hybrid book in Lemma 8 and the corresponding  $\beta_j$ 's and  $\theta_j^{DHM}$ 's in Equations (23) and (25). This is best understood in terms of limit order execution probabilities. Consider an infinitesimally small PLM limit sell  $S_j^p = \epsilon$  at a price  $p_j < p_{\max}$ . Only infinitesimal deviations need to be considered since value traders are individually negligible. As illustrated in Figure 6, the limiting execution probability  $\lim_{\epsilon \rightarrow 0} \Pr(B^p > \epsilon)$  given an otherwise empty PLM book is the sum of the probabilities that  $\epsilon$  is used as a pressure valve—that is, when the total volume  $x$  is in an interval  $(\theta_k^{DHM}, \beta_k)$  for some price  $p_k \leq p_j$ —plus the probability that  $x > \theta_{j+1}^h$  so that both the HM and the hypothetical PLM limit sells at  $p_j$  execute. In contrast, the probability that the marginal HM limit order executes is simply the probability that  $x \geq \beta_j$ . Differencing the two probabilities gives another restriction on the volume distribution  $F$ :

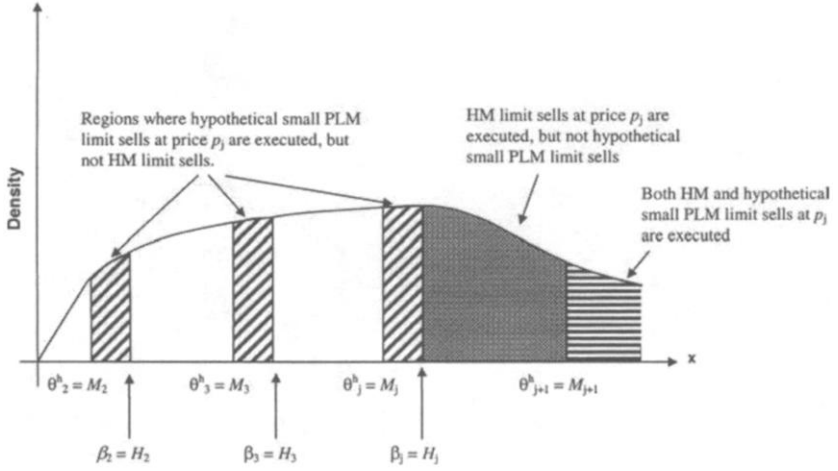
**Lemma 10.** *A second necessary condition for a dominant HM equilibrium with HM preferencing is*

$$\Pr(H_j \leq x \leq M_{j+1}) - \sum_{p_k \leq p_j} \Pr(M_k < x < H_k) \geq 0 \quad (27)$$

at prices  $p_j = p_{\min}, \dots, p_{\max-2}$  and

$$\frac{c_{\max}/\alpha}{p_{\max} - v} \geq \sum_{p_k \leq p_{\max-1}} \Pr(M_k < x < H_k). \quad (28)$$

Only one restriction, Inequality (28), is needed for both  $p_{\max-1}$  and  $p_{\max}$ . Since the hypothetical execution probability at  $p_{\max}$  is the same as at  $p_{\max-1}$ —both



**Figure 6**  
Regions for PLM and HM marginal limit order execution at price  $p_j$  with HM preferencing

limit orders only execute in their pressure valve roles given HM preferencing—if PLM limit orders at  $p_{\max}$  are not profitable, then neither are those at  $p_{\max-1}$ .

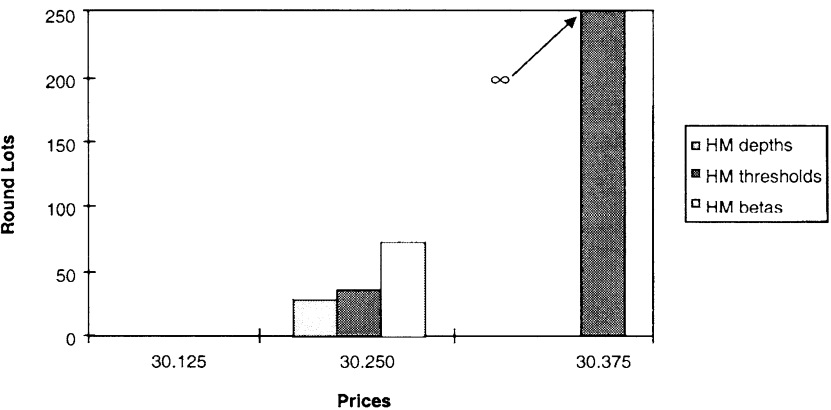
In addition to being necessary, Lemmas 9 and 10 are also, by construction, sufficient for a dominant HM equilibrium.

**Proposition 3.** *Any continuous, strictly increasing distribution  $F$  which satisfies Lemmas 9 and 10 supports a dominant hybrid market equilibrium.*

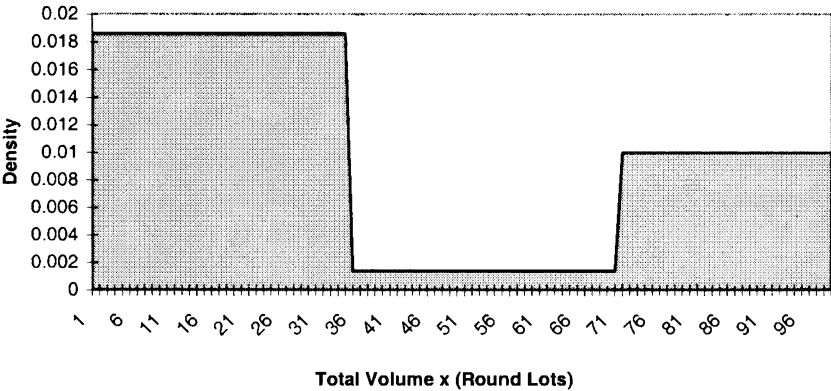
We denote the set of distributions that satisfy Conditions 1 and 2 for a given market environment  $\sigma = \{\mathcal{P}, c_{\min}, \dots, \alpha, r\}$  as  $\mathcal{F}^\sigma$ . The quantities  $H_j$  and  $M_j$  implicitly describe key features of the distribution of the random demand for liquidity  $F$ . However, what do these distributions look like more intuitively? And do such distributions exist for any environment  $\sigma$ ?

When  $p_{\max} > p_{\min} \geq p_2$ , we can always find *some* volume distributions which satisfy Conditions 1 and 2. First,  $F$  must be skewed enough to make each of the  $M_{j+1}$ 's (weighted averages of the  $H_{j+1}$ 's and lower  $H_k$ 's) larger than the  $H_j$ 's as required by Lemma 9. Second, we need enough “clustering” to ensure that the conditions in Lemma 10 hold. In particular, although the overall probability  $\Pr(H_j \leq x < H_{j+1})$  in each interval  $[H_j, H_{j+1})$  is fixed at  $\frac{c_j/\alpha}{p_j-v} - \frac{c_{j+1}/\alpha}{p_{j+1}-v}$  by Equation (13), we need a sufficiently *large* probability in the bottom subintervals  $[H_j, M_{j+1}]$  and a sufficiently *small* probability in the corresponding top subintervals  $(M_{j+1}, H_{j+1})$  to satisfy Lemma 10. From the preceding discussion it follows that  $\mathcal{F}^\sigma$  is nonempty.

Figure 7a illustrates a dominant HM equilibrium. The parameterization is the same as in Figures 4 and 5 with one difference. As illustrated in Figure 7b, the distribution  $F$  here differs from the previous uniform  $[0, 100]$



A: DHM depths, thresholds, and betas



B: Total volume density supporting a DHM with HM preferencing

**Figure 7**  
**A dominant hybrid market equilibrium with HM preferencing**  
Parameter values: common value  $v = \$30.09$ , ex ante limit order submission costs  $c_1 = \$0.0263$ ,  $c_2 = \$0.0225$ ,  $c_3 = \$0.0188$ , probability of a buy  $\alpha = 0.5$ , and  $p_{\max} = \$30.375$ . The parameters are the same as in the other examples except that the total volume distribution  $F$  in Figure 7B is not uniform.

distribution in that the probability  $\Pr(0 < x \leq M_2 = 35.9)$  is increased to 67% to offset a reduction in the probability  $\Pr(M_2 < x < H_2 = 71.9)$  to 5%. This is an example of the probability clustering described in Lemma 10.<sup>16</sup> The execution threshold  $\theta_2^h$  for hybrid limit sells at \$30.25 is roughly 35.9 round lots. As Figures 7c and 7d illustrate, for volumes between  $\theta_2^h$  and  $\beta_2 = 71.9$

<sup>16</sup> With a uniform distribution for total volume, the execution probabilities for infinitesimal PLM limit orders at  $p_2$  and  $p_3$  are both  $\Pr(M_2 < x < H_2) = 36\%$  which exceeds their respective break-even probabilities  $\frac{c_2/\alpha}{p_2-v} = 28\%$  and  $\frac{c_3/\alpha}{p_3-v} = 13\%$ . The violation of Condition 2 in the uniform example is not caused by the uniform distribution per se, but rather by this combination of parameters taken together. For example, if  $\alpha$  is reduced to 25%, then the uniform distribution does support a DHM equilibrium.

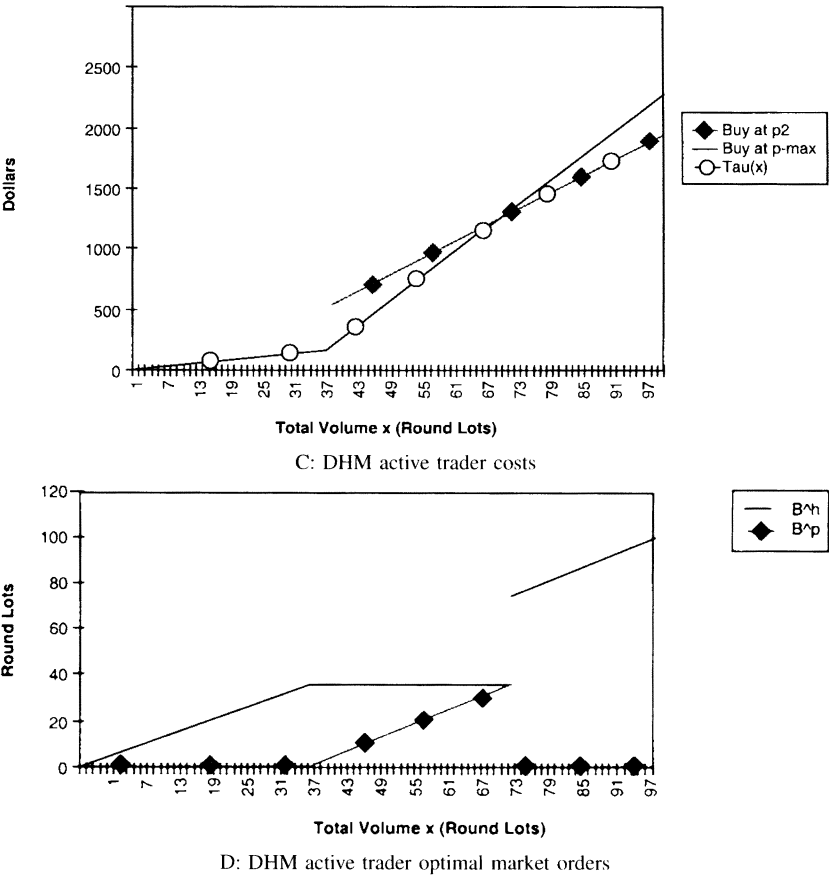


Figure 7  
(continued)

round lots the cost  $T^h(\theta_2^h) + T^p(x - \theta_2^h)$  of using the PLM crowd as a pressure valve is less than the cost of buying the full  $x$  shares outright in the hybrid market. Thus in the interval  $(35.9, 71.9)$ , the hybrid order  $B^h$  is constant at 35.9 round lots which, in turn, implies that the specialist's profit  $\pi$  over this region is constant at \$125.8 and that the active trader's total cost schedule  $\tau(x)$  has a steep slope equal to  $p_{\max} - v = \$0.285$ . Above 71.9 round lots, however, it is cheaper to buy the whole  $x$  shares in the HM. At this point the slope of the active trader's cost schedule  $\tau(x)$  falls from  $p_3 - v$  to  $p_2 - v = \$0.16$  and the specialist's profit jumps discontinuously and then is (once again) increasing in  $x$ .

The comparative statics for local changes in  $\alpha$ ,  $r$ , and  $c_j$ —that is, for which  $F$  is still in the new set  $\mathcal{F}^\sigma$ —are again intuitive. Increased demand for sell liquidity and lower submission costs lead to a deeper DHM book and greater aggregate liquidity. Unfortunately comparative statics for the set

$\mathcal{F}^\sigma$  itself are ambiguous since changing  $\alpha$ ,  $r$ , and  $c_j$  involves checking the conditions in Lemmas 9 and 10 for different regions of the distribution  $F$ . We do have a comparative static result for changes in the common price grid  $\mathcal{P}$  when the tick size is a constant  $p_j - p_{j-1} = \Delta$ .

**Proposition 4.** *Holding the distribution  $F$  and the rest of the market environment  $\sigma$  fixed, it is not possible to support a DHM equilibrium if the tick size  $\Delta$  is too small.*

One implication of this result is that the recent move to decimal pricing in the United States may erode the dominance of the (hybrid) NYSE in equity trading in the future. Holthausen, Leftwich, and Mayers (1987) and others estimate that the transitory premium/discount on block trades, a natural proxy for  $p_{\max} - v$ , was rarely more than a couple of ticks before decimalization. This suggests that a DHM equilibrium was quite natural then, since Conditions 1 and 2 are easier to satisfy on a coarse price grid. However, in the new postdecimalization regime, with much finer price grids, this may no longer be true.

Having shown that pure limit order markets are not exclusively competition-proof when  $F \in \mathcal{F}^\sigma$ , we next show that liquidity need not concentrate in a single market. In particular, coexistence of competing pure and hybrid books is also possible.

**Proposition 5.** *If  $p_{\max} > p_{\min}$ , then hybrid preferencing leads to coexistence of both the PLM and HM limit order books for distributions  $F \notin \mathcal{F}^\sigma$ .*

Coexistence equilibria with HM preferencing can be computed recursively in the special case where  $p_{\max} = p_3$ . Using our maintained assumption that  $p_{\min} > p_1$  and thus  $S_1^h = S_1^p = 0$ , we solve the equation  $T^h(\beta_2) + T^p(0) = T^h(\theta_2^h) + T^p(\beta_2 - \theta_2^h)$  at  $p_2$  in the hybrid market to get the critical value

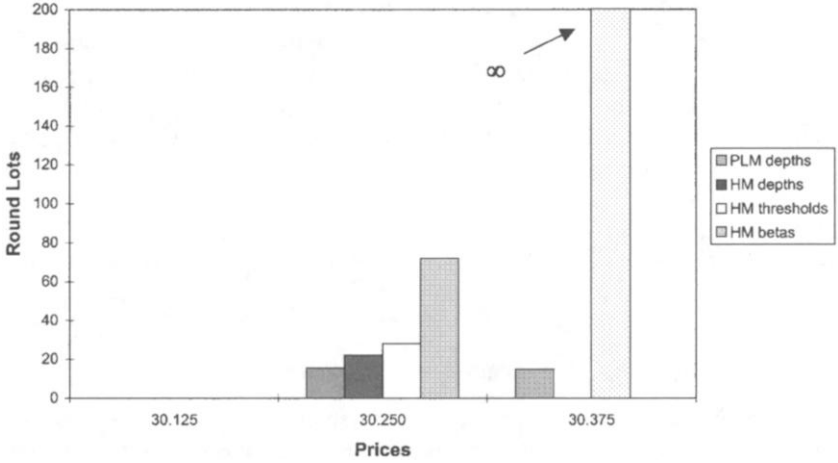
$$\beta_2 = \frac{\psi_2}{\gamma_2} S_2^h + S_2^p \quad (29)$$

and then invert  $\Pr(B^h > \theta_2^h) = \Pr(x \geq \beta_2) = \frac{c_2/\alpha}{p_2 - v}$  to get the hybrid depth and the associated execution threshold

$$S_2^h = (H_2 - S_2^p) \frac{\gamma_2}{\psi_2} \quad \text{and} \quad \theta_2^h = \frac{H_2 - S_2^p}{\psi_2}. \quad (30)$$

Given HM preferencing and  $p_{\max-1} = p_2$ , the active trader buys everything in the hybrid market (i.e.,  $B^p = 0$ ) once  $x \geq \beta_2 = H_2$ . Thus the PLM depth  $S_2^p$  at  $p_2$  (if nonnegative) can be computed numerically from

$$\Pr(B^p \geq Q_2^p) = \Pr\left(\theta_2^h + Q_2^p = \frac{H_2 - S_2^p}{\psi_2} + S_2^p \leq x < H_2\right) = \frac{c_2/\alpha}{p_2 - v}. \quad (31)$$



**Figure 8**  
**Example of coexistence with HM preferencing**

Parameter values: common value  $v = \$30.09$ , ex ante limit order submission costs  $c_1 = \$0.0263$ ,  $c_2 = \$0.0225$ ,  $c_3 = \$0.0188$ , probability of a buy  $\alpha = 0.5$ ,  $p_{\max} = \$30.375$ , volume  $|x|$  uniform over  $[0, 100]$ . See Figure 4 for strategies, cost schedules, and order arrival distributions.

Given the solution for  $S_2^p$ , we then compute  $S_3^p$  at  $p_3$  from

$$\begin{aligned} \Pr(B^p \geq Q_3^p) &= \Pr\left(\theta_2^h + Q_3^p = \frac{H_2 - S_2^p}{\psi_2} + S_2^p + S_3^p \leq x < H_2\right) \\ &= \frac{c_3/\alpha}{p_3 - v}. \end{aligned} \quad (32)$$

Figure 8 gives the limit order books and thresholds for the numerical example of coexistence illustrated earlier in Figure 4. Since Condition 2 is not satisfied for a uniform distribution  $F$  and these parameters (see note 16), both limit order books have positive depths. These books and the specialist's optimal response lead, in turn, to the equilibrium liquidity supply schedules depicted in Figure 4.

Unfortunately when  $p_{\max} > p_3$ , the equilibrium with HM preferencing cannot be constructed recursively when  $F$  is not in  $\mathcal{F}^\sigma$ . This is because the possibility of positive depths at higher prices in the PLM book now affects the profitability of limit sells at lower prices.<sup>17</sup>

## 2.4 Other preferencing rules

The two extreme rules considered here highlight the range of possible equilibrium allocations of liquidity and trading across the pure and hybrid markets. Clearly many other preferencing rules—for example, with deterministic

<sup>17</sup> None of this changes the fact that, with PLM preferencing, a dominant PLM equilibrium always exists and can be calculated recursively as above. Similarly, as long as  $F \in \mathcal{F}^\sigma$ , a dominant HM equilibrium with HM preferencing also exists and can still be calculated recursively. Nonrecursivity is only an issue outside of these cases.



or  $x$ -contingent pro rata sharing and/or randomization—are also possible. Without analyzing the details of such alternatives, we make a few general observations. First, two preferencing rules which differ only in the partial execution of inframarginal limit orders in the PLM book have the same marginal break-even conditions  $e_j^p \leq 0$  and  $e_j^h \leq 0$ , and hence the identical equilibrium depths. Second, preferencing rules which are intermediate between HM and PLM preferencing should have intermediate equilibrium depths. Thus many of these intermediate preferencing rules will also support coexistence of the pure and hybrid limit order books.

3. Comparisons with Other Trading Architectures

When regulators decide whether to permit new trading venues to open, they are, naturally, concerned about the possible fragmentation of order flow and any impact it might have on aggregate liquidity. Corporations have similar concerns in deciding whether to cross-list their stock in multiple markets. Easley, Kiefer, and O'Hara (1996) and Battalio and Holden (1996) focus on price discovery and intermarket “cream skimming,” whereas we focus on the impact of intermarket competition on aggregate liquidity. We consider several alternative institutional arrangements and compare the resulting liquidity supply and trading outcomes. We find significant differences depending on the institutional arrangement of the market. Table 1 summarizes the results discussed below.

3.1 Competition versus a single hybrid market

Consider a stock which initially trades on just the NYSE (i.e., a hybrid market). Would the opening of an ECN (i.e., a pure limit order market) improve aggregate liquidity?

Before the ECN opens the active trader has no choice about where to trade and hence  $B^h = x$ . This leads to the single hybrid market (SHM) equilibrium

Table 1  
Welfare comparisons across equilibria for active traders with different total volumes

Pairing	Small volumes	Mid-size volumes	Large volumes
SPLM versus DPLM	Indifferent	Indifferent	DPLM
SHM versus DHM	Indifferent	SHM	SHM
SHM (or DHM) versus DPLM	SHM (or DHM)	Ambiguous	DPLM
SPLM versus DHM	DHM	Ambiguous	DHM

from Seppi (1997) in which the hybrid book and corresponding execution thresholds are given by

$$S_j^{SHM} = (H_j - Q_{j-1}^{SHM})\gamma_j \quad \text{at prices } p_j = p_{\min}, \dots, p_{\max-1} \quad (33)$$

$$\theta_j^{SHM} = \begin{cases} H_j & \text{at prices } p_j = p_{\min}, \dots, p_{\max-1} \\ \infty & \text{at } p_{\max}. \end{cases} \quad (34)$$

Who wins and loses after the ECN opens depends on which competitive equilibrium prevails. If a dominant HM equilibrium prevails, then small retail investors (with small  $x$ 's) are indifferent while medium and large institutional investors (for whom  $x$  is larger) are worse off. In both the SHM and DHM equilibria the specialist provides liquidity at  $p_{\min-1}$  on small orders and at  $p_{\max-1}$  on very large orders, but the DHM limit order book is less deep than the SHM book. In particular, comparing the depths in Equation (24) with the  $S_j^{SHM}$ 's leads to the following:

**Proposition 6.** *For every  $F \in \mathcal{F}^\sigma$ , the cumulative single HM depth  $Q_j^{SHM}$  at each price  $p_j \geq p_{\min}$  is greater than the corresponding depth  $Q_j^{DHM}$  in the dominant HM equilibrium.*

Intermarket competition and the resulting order flow fragmentation actually reduce aggregate liquidity in this case. The first reason is that the lower cumulative depth in the DHM book reduces the inframarginal liquidity from limit orders at prices below  $p^h$  when  $x > \beta_{\min+1} = H_{\min+1}$ . The second is that when  $\theta_j^{DHM} = M_j < x < H_j = \beta_j^{DHM} = \theta_j^{SHM}$ , the active trader pays  $p_{\max}$  (in the pure market) on her last  $x - M_j$  shares in the DHM equilibrium, whereas she pays no more than  $p_{j-1}$  in the SHM equilibrium. Thus, although using the PLM crowd as a pressure valve reduces the active trader's cost *taking the HM book as given*, once the equilibrium effect on the endogenous depths is included—that is, after taking into account the response of value traders who rationally anticipate lower HM limit order execution probabilities due to order flow fragmentation—the DHM equilibrium has higher trading costs.

A dominant HM equilibrium is not the only possible competitive outcome. If a dominant PLM equilibrium prevails instead (i.e., if the ECN attracts all of the limit orders and most of the market order flow), then the comparison with the single HM equilibrium is ambiguous. Large traders, with  $x > H_{\max-1}$ , now prefer competition and the dominant PLM equilibrium since the maximum cleanup prices are the same (i.e.,  $p_{\max-1}$ ) and the DPLM book is deeper (i.e.,  $Q_j^{SHM} < \theta_j^{SHM} = H_j = Q_j^{DPLM}$ ). In contrast, small retail investors, for whom  $x < H_{\min}$ , strictly prefer a single HM (or the DHM) equilibrium because of the intraspread price improvement the specialist provides to undercut the HM book at  $p_{\min} > p_1$  on small trades. This is not a minor consideration. Ross, Shapiro, and Smith (1996) find that intraspread execution by the specialist accounts for one-quarter of the price improvement on the NYSE. Barclay,

Hendershott, and McCormick (2001) also find lower price impact costs on small dealer-executed trades than on small ECN trades on the Nasdaq.

### 3.2 Competition versus a single pure limit order market

Now consider the entry of a hybrid market when the stock is initially listed only on a pure limit order market. Once again, there are a multiplicity of possible post-entry competitive equilibria depending on the preferencing rule. The trick with this comparison, however, is to describe the correct *initial* equilibrium with the pure limit order market alone. In addition to the active trader, value traders, and the PLM trading crowd, there is also the person who would otherwise be the HM specialist to consider.

We consider the simplest case by assuming that the specialist's cost advantage derives entirely from his privileged institutional position in the hybrid market.<sup>18</sup> Thus, in a single PLM setting without these institutional advantages, he is just another dealer in the crowd with costs  $r$ . In this case, Seppi's (1997) SPLM equilibrium—in which  $B^p = x$  and  $S_j^{SPLM} = H_j - H_{j-1}$  up through  $p_{\max}$  (i.e., rather than just  $p_{\max-1}$  as in the DPLM)—is the correct pre-entry reference point. If a DHM equilibrium prevails after the hybrid market enters, then the comparison with the SPLM is ambiguous. Very small and very large investors prefer the DHM since they value the price improvement from the specialist on small trades (undercutting the book at  $p_{\min-1}$ ) and on large trades (undercutting the crowd at  $p_{\max-1}$ ). Intermediate investors benefit from the deeper limit order book in the SPLM. If instead a DPLM equilibrium prevails after the hybrid market enters, then aggregate liquidity is unambiguously better. In particular, liquidity is unchanged for small and midsize orders (since the SPLM and DPLM books are identical up through  $p_{\max-1}$ ) and liquidity is strictly better in the DPLM for large trades (since the specialist again provides infinite liquidity at  $p_{\max-1}$ ).

### 3.3 Liquidity providers' welfare

One reason to focus on the cost of liquidity to the active trader is that liquidity is empirically observable. This is not, however, to say that the liquidity providers' welfare is unimportant. One immediate implication of our analysis is that value traders are better off in equilibria in which limit orders are placed in the PLM book (e.g., SPLM, DPLM, and coexistence) than in equilibria in which limit orders are posted only in the HM book (e.g., the SHM and DHM). Although the marginal limit orders just break even in either market, the *inframarginal* limit orders at prices  $p_j > p_1$  earn positive expected profits in the pure limit order market due to the possibility of partial execution. In contrast, recall that limit orders at  $p_j > p_1$  in the hybrid market are either executed in toto or not at all. Thus there is no profitable inframarginal HM order execution above  $p_1$ . Conversely the specialist clearly trades more and is thus better off in the SHM and DHM equilibria than in the DPLM.

<sup>18</sup> Details about comparisons with other pre-entry assumptions are available from the authors.

### 3.4 Endogenizing the total trading volume

One caveat about our welfare comparisons is that the total volume distribution  $F$  is exogenous. Ideally the demand for liquidity  $x$  and its distribution  $F$  should be modeled as functions of the prevailing liquidity supply schedules  $T^h$  and  $T^p$ . In particular, discontinuities (as in the SHM equilibrium) and nonconvexities (as in the DHM equilibrium) in the aggregate liquidity supply schedule  $\tau(x)$  mean that active traders with any discretion in their total trading needs will sometimes curtail their trading in order to optimize the trade-off between trading costs and the amount traded (e.g., to avoid a discontinuity). We leave this important topic for future work.

### 3.5 “Best execution” regulations

When exchanges compete under a common regulatory umbrella, such as two markets in the same country, regulators can and do impose cross-exchange “best execution” restrictions on pricing [see Macey and O’Hara (1997)]. The creation of the National Market System in 1975 and the 1996 SEC “order handling” rules protecting customer limit orders on the Nasdaq are two such examples. More recently the controversy over a centralized limit order book was, in part, a debate about whether to impose cross-market time priority and whether to permit ex post price improvement by specialists [see Levitt (2000) and NYSE (2000)]. Such regulatory interventions directly affect the incentives of liquidity providers. We study these issues by varying the institutional environment and seeing how this affects the behavior of our four types of investors.

If price, time, and public order priority are imposed uniformly across all exchanges, then the result is a single integrated market. In our model, with a specialist, this leads to the SHM equilibrium. Our intent here is to explore less sweeping cross-market regulations between the polar cases of unregulated competition (as in Section 2) and a de facto single integrated market.

A natural intermediate case to consider is one in which price priority applies globally across markets, but where time and public order priority are still local. One way to model such a regime is to view global price priority as a constraint on order execution. In particular, if the specialist wants to clean up at  $p_j$  when  $B^p < Q_{j-1}^p$ , then he must first redirect  $Q_{j-1}^p - B^p$  shares from  $B^h$  over to the PLM to clean up the unexecuted limit sells through  $p_{j-1}$ . The PLM limit sells at  $p_j$ , however, may stay unfilled. Similarly the PLM order execution system must divert orders to the hybrid market if there are unexecuted HM limit orders at  $p_{j-1}$  once all of the PLM limit sells at  $p_{j-1}$  are exhausted. Unfortunately, while such a representation is realistic, it quickly leads to dynamic interactions between the two markets which are beyond our static model. To avoid these complications we instead impose global price priority as a constraint on the active trader’s order submission problem.

**Definition 6.** Global price priority requires the active trader to submit orders,  $B^p$  and  $B^h$ , such that the corresponding market clearing prices,  $p^h$  and  $p^p$ , are no more than one “tick” apart. Thus, given ordered thresholds and depths  $\theta_j^h < \theta_{j+1}^h$  and  $Q_{j-1}^p \leq Q_{j+1}^p$ , the orders  $B^h$  and  $B^p$  must jointly satisfy  $\theta_j^h < B^h \leq \theta_{j+1}^h$  and  $Q_{j-1}^p \leq B^p \leq Q_{j+1}^p$ .

Giving price priority to the PLM limit orders vis-à-vis the HM specialist has little practical “bite.” Recall from Lemma 4 that, even without such a constraint, the optimal PLM order is  $B^p \geq Q_{j-1}^p$  when  $p^h = p_j$ . Applying global price priority in the other direction, however, dramatically curtails the PLM’s pressure valve role. In particular, the active trader can no longer buy at  $p^p \geq p_{j+2}$  in the PLM so as to keep the specialist at  $p^h = p_j$ , since doing so would leave the HM limit sells at  $p_{j+1}$  unfilled. In this direction, global price priority does have some “bite.”

**Proposition 7.** Given global price priority across markets, hybrid market preferencing leads to a single hybrid market (SHM) equilibrium.

**Proposition 8.** Multiple equilibria can be supported with PLM preferencing and global price priority in that the two markets’ books can be either (a) the dominant PLM equilibrium books with  $S_j^p = S_j^{DPLM}$  and  $S_j^h = 0$  or (b) any set of nonnegative HM depths  $S_2^h, \dots, S_{\max-1}^h$  and the corresponding PLM depths  $S_2^p, \dots, S_{\max-1}^p$  satisfying

$$S_j^h / \gamma_j + S_j^p = H_j - Q_{j-1}^h - Q_{j-1}^p \quad (35)$$

$$S_j^h > \gamma_j \left[ \frac{1}{\gamma_{j-1}} - 1 \right] S_{j-1}^h. \quad (36)$$

The net effect of global price priority is a dramatic improvement in the competitive viability of hybrid markets regardless of the preferencing rule. Proposition 7 shows that an SHM equilibrium can now be sustained for any distribution  $F$  rather than just those in  $\mathcal{F}^\sigma$ . Even with PLM preferencing, Proposition 8 shows that equilibria with positive depths in the HM book—including an SHM equilibrium—are now possible in addition to the DPLM equilibrium.

#### 4. Conclusion

We have modeled liquidity-based competition for market and limit order flow in an environment in which liquidity providers have heterogeneous costs. Multiple equilibria exist, depending on the preferencing rule used by investors to decide, when indifferent, where to route their market orders. Some of the equilibria involve coexistence of competing exchanges while others have concentrated trading and liquidity on a single exchange. Small

“tie-breakers” such as “payment for order flow,” habits, etc., can have a significant impact on equilibrium selection. We also show that “best execution” regulations such as global price priority have real “bite” in that they prevent cross-market marginal price differences of *two* ticks or more. This reduces the ability of a pure limit order market to act as a pressure valve, thereby dramatically increasing the competitive viability of a hybrid limit order/specialist market. We see many promising directions for future work. These include

- Characterizing the relationship between preferencing and the robustness of coexistence versus dominant market equilibria more systematically. We have only considered two polar extremes.
- Introducing asymmetries across markets such as different ex ante limit order submission costs  $c_j$ , crowd reservation profit levels  $r$ , or tick sizes. Letting markets independently choose their own price grids would be particularly interesting.
- Adding exchange-specific clienteles of investors who have different relative costs for submitting limit and/or market orders on the competing exchanges.
- Endogenizing the total volume distribution  $F$  to allow for a trade-off between trading costs and the total number of shares traded.

## Appendix

*Proof of Lemma 1.* This follows from the continuity of  $F$  together with the continuity of  $T^h$  in  $S_j^h$  (i.e., since the thresholds  $\theta_j^h$  are continuous in the HM depths), the continuity of  $T^p$  in  $S_j^p$  and thus the respective continuity of  $B^h$  and  $B^p$ .

*Proof of Lemmas 2 and 3.* These follow from the same logic as Seppi’s (1997) Proposition 2, given that the equilibrium HM book satisfies a break-even condition  $e_j^p = 0$  at each price with positive depth  $S_j^h > 0$ .

*Proof of Lemma 4.* Consider  $p_j > p_1$ . If  $p^h[B^h(x)] = p_j$  and  $B^p(x) < Q_{j-1}^p$ , then the active trader could lower her total trading costs by reducing the number of shares she buys at  $p_j$  in the hybrid market and increasing her order to the PLM where she can still buy at  $p^p \leq p_{j-1} < p_j = p^h$ . Given continuity of  $T^p$ , buying more in the PLM does not affect the price paid on the  $B^p(x)$  shares currently being bought there. Thus, in equilibrium,  $B^p(x) \geq Q_{j-1}^p$  and  $p^p \geq p_{j-1}$  whenever  $p^h = p_j$ .

*Proof of Lemma 5.* Consider  $\underline{x}_j^h = \inf\{x | B^h(x) > \theta_j^h\}$  and  $\underline{x}_j^p = \inf\{x | B^p(x) \geq Q_j^p\}$  at any price  $p_{\max} > p_j > p_1$ , where  $S_j^h, S_j^p > 0$ . The lemma asserts that  $\underline{x}_j^h > \underline{x}_j^p$ . Notice that  $\min\{x_j^h, x_j^p\} \leq \theta_j^h + Q_j^p$ , since  $x = \theta_j^h + Q_j^p$  is a strict upper bound on how many shares can be bought with both  $B^h \leq \theta_j^h$  and  $B^p < Q_j^p$ . Thus, consider volumes  $x \leq \theta_j^h + Q_j^p$ . For such volumes, buying at  $p^p = p_j$  in the PLM strictly dominates buying at  $p^h = p_j$  in the hybrid market, since the hybrid liquidity cost schedule  $T^h$  has a discontinuity at  $\theta_j^h$ , whereas the PLM cost schedule  $T^p$  has a “kink” at  $Q_j^p$  but is continuous. Thus  $\underline{x}_j^p \leq \theta_j^h + Q_j^p < \underline{x}_j^h$ .

*Proof of Proposition 1.* By construction, the equilibrium conditions  $e_j^p \leq 0$  are satisfied in the PLM when the HM book is empty. In addition,  $B^p$  and  $B^h$  are clearly optimal given the empty

HM book and the presence of a specialist willing to sell at  $p_{\max-1}$ . Thus we just need to show that  $e_j^h \leq 0$  with an empty HM book. This follows directly from Inequality (18) and the observation that, with PLM preferencing, the  $\Pr(S_j^h \text{ executes, but not } S_j^p \text{ due to preferencing}) = 0$  given any  $S_j^h > 0$ , in which case  $e_j^h < e_j^p \leq 0$ .

*Proof of Proposition 2.* Suppose the active trader were to deviate from the DPLM market order strategy and posted an HM limit buy for  $y$  shares at  $p_j < p_{\max-1}$  in an attempt to elicit liquidity from the specialist below  $p_{\max-1}$ . Recall that the specialist cannot see and, hence, condition on the realized PLM order  $B^p$ . Seeing an HM limit buy  $y$ , the specialist cannot tell whether the active trader is trying to buy a total of  $x = Q_{\max-1}^p + y$  shares or  $x = Q_j^p + y < Q_{\max-1}^p$  shares across the two markets. In the former case, the specialist would optimally leave  $y$  unexecuted, thereby forcing the active trader to resubmit a *market* buy for  $y$  (on which the specialist receives the full  $p_{\max-1}$ ). In the latter case, however, he would go ahead and execute  $y$ , since otherwise the active trader will be forced to buy  $y$  at  $p_{j+1}$  from the PLM book and the specialist will get nothing. Since such a limit buy  $y$  is an off-equilibrium event—in a DPLM equilibrium there are no HM limit buys above  $v$ —we are free to support the DPLM equilibrium by giving the specialist sufficiently suspicious off-equilibrium beliefs in which the probability of the first case ( $x = Q_{\max-1}^p + y$ ) is enough greater than the probability of the second ( $x < Q_{\max-1}^p$ ) to make him reject limit buys above  $v$ . Thus our DPLM equilibrium is Bayesian Nash with regard to this expanded active trader action space.

*Proof of Lemma 6.* Consider two possible shocks  $x$  and  $X > x$  and any two pairs of associated optimal orders  $(b^h, b^p)$  and  $(B^h, B^p)$ . Suppose the lemma is not true, so that  $B^h < b^h$ . We can rewrite the total cost of buying  $X$  shares via orders  $(B^h, B^p = X - B^h)$  as  $T^h(B^h) + T^p(x - B^h) + [T^p(X - B^h) - T^p(x - B^h)]$ , where  $T^h(B^h) + T^p(x - B^h)$  is the cost of buying  $x < X$  shares via orders  $(B^h, x - B^h)$  and  $T^p(X - B^h) - T^p(x - B^h)$  is the incremental cost of buying an additional  $X - x$  shares in the PLM (i.e., in addition to the  $x - B^h$  shares) to bring the total shares bought to  $X$ . Next, compare this with the cost of buying  $X$  shares via an alternative pair of orders  $(b^h, X - b^h)$ , which we write as  $T^h(b^h) + T^p(x - b^h) + [T^p(X - b^h) - T^p(x - b^h)]$ . First, note that if  $(b^h, b^p)$  is optimal when buying  $x$  shares, then  $T^h(B^h) + T^p(x - B^h) \geq T^h(b^h) + T^p(x - b^h)$ . Second, note that  $X > x$  and the hypothesis  $B^h < b^h$  implies  $x - B^h > x - b^h$ , and thus that  $T^p(X - B^h) - T^p(x - B^h)$  is greater than  $T^p(X - b^h) - T^p(x - b^h)$ . Taken together these two observations imply, however, that it is cheaper to buy  $X$  shares via  $(b^h, X - b^h)$  than via  $(B^h, X - B^h)$  which is a contradiction. Thus we must have  $B^h \geq b^h$ .

*Proof of Lemma 7.* The monotonicity of  $B^h$  in  $x$ , together with HM preferencing, implies a minimum critical volume  $\beta_j$  exists such that  $p^h[B^h(x)] \geq p_j$  when  $x \geq \beta_j$ . Furthermore, the equilibrium  $\beta_j$  must be strictly less than  $\beta_{j+1}$ , since if  $\beta_j = \beta_{j+1}$ , then  $0 = e_j^h < e_{j+1}^h$ , which is a contradiction.

The expression in Equation (20) of the lemma asserts that  $\beta_j$  solves  $T^h(\beta_j) + T^p(0) = T^h(\theta_j^h) + T^p(\beta_j - \theta_j^h)$ , as in Equation (22). In particular, this assumes that the orders  $B^h(\beta_j) = \beta_j$  and  $B^p(\beta_j) = 0$  are cost minimizing when buying  $x = \beta_j$  shares. Suppose not, so that  $B^h(\beta_j) < \beta_j$  and  $B^p(\beta_j) = \beta_j - B^h(\beta_j) > 0$ . Notice, however, that if orders  $(B^h(\beta_j), \beta_j - B^h(\beta_j))$  minimize the cost of buying  $x = \beta_j$  shares, then the orders  $(B^h(\beta_j), 0)$  must minimize the cost of buying  $x^* = B^h(\beta_j)$  shares. Otherwise, if some alternative pair of orders  $(b^*, x^* - b^*)$  minimizes the cost of buying  $x^*$ , then it would be cheaper to buy  $x = \beta_j$  shares via  $(b^*, \beta_j - x^* + x^* - b^*)$  than via  $(B^h(\beta_j), \beta_j - B^h(\beta_j))$ , since in either case the last  $\beta_j - x^*$  shares are bought at  $p_{\max}$  from the PLM crowd. Thus  $(B^h(\beta_j), 0)$  must minimize the cost of buying  $x^* = B^h(\beta_j)$  shares. However, this contradicts the initial hypothesis that our candidate  $\beta_j$  is the minimum  $x$  such that  $p^h = p_j$ .

*Proof of Lemma 8.* Substituting  $\Pr(B^h > \theta_j^h) = \Pr(x \geq \beta_j)$  into Equation (12) and then inverting and recursively rearranging using Equation (20) gives the result.

*Proof of Lemma 9.* If  $H_j > M_{j+1}$ , then  $\beta_j > \theta_{j+1}^{DHM}$ , and the specialist is not willing to provide enough liquidity at  $p_j$  for the postulated strategy of the active trader to be feasible.

*Proof of Lemma 10.* The conditions in the lemma ensure that small deviations from an empty PLM book are not profitable. In a DHM equilibrium the probability of execution of an infinitesimal PLM limit order  $\epsilon > 0$  (i.e., a deviation from  $S_j^p = 0$ ) at  $p_j \geq p_{\min} > p_1$  must always be less than the break-even execution probability of the marginal HM limit order at  $p_j$ . Letting  $\epsilon \rightarrow 0$ , the probability that a hybrid limit order is executed (in the DHM) is

$$\Pr(B^h > \theta_j^h) \rightarrow \Pr(x \geq \beta_j = H_j). \quad (37)$$

The corresponding probability of execution of  $\epsilon$  in the PLM is

$$\Pr(B^p > \epsilon) \rightarrow \sum_{p_k \leq p_j} \Pr(M_k < x < H_k) + \Pr(x > \theta_{j+1}^h = M_{j+1}), \quad (38)$$

where the first term is the total probability of a hypothetical infinitesimal PLM limit sell at  $p_j$  being executed in a pressure valve role (i.e., when keeping  $B^h = \theta_k^h$  at some  $p_k \leq p_j$ ) and the second term is the probability that both the HM and PLM limit sells at  $p_j$  are executed. Subtracting Probability (38) from Probability (37) and canceling terms yields Inequality (27) at prices below  $p_{\max-1}$ .

Prices  $p_{\max}$  and  $p_{\max-1}$  are special cases. Since the specialist provides unlimited liquidity at  $p_{\max-1}$  (i.e.,  $\theta_{\max}^h = \infty$ ), the limiting probability of execution for  $\epsilon$  PLM limit orders at  $p_{\max}$  is  $\sum_{p_k \leq p_{\max-1}} \Pr(M_k < x < H_k)$ . In a DHM equilibrium this probability must be less than  $\frac{c_{\max}/\alpha}{p_{\max}-v}$  (so that  $S_{\max}^p = 0$ ), which gives the condition in Inequality (28). Notice, further, that with HM preferencing, the execution probability for PLM limit sells at  $p_{\max-1}$  is the same as at  $p_{\max}$  so that if Inequality (28) holds, then  $e_{\max-1}^p < e_{\max}^p \leq 0$  and  $S_{\max-1}^p = 0$  automatically.

*Proof of Proposition 3.* This follows directly from the arguments in the body of the article.

*Proof of Proposition 4.* We prove this by showing that Condition 1 for a DHM equilibrium is eventually violated as the (constant) tick size  $\Delta$  gets sufficiently small. In particular, consider  $p_{\max-2}^\Delta$ , the second highest price below  $v+r$  given a particular tick size  $\Delta$ . Condition 1 requires that  $\theta_{\max-1}^h = M_{\max-1} = \frac{1}{\psi_{\max-1}} H_{\max-1} + (1 - \frac{1}{\psi_{\max-1}}) Q_{\max-2}$  must be strictly greater than  $\beta_{\max-2} = H_{\max-2}$ . To see that this is not possible as  $\Delta \rightarrow 0$ , notice first that  $\frac{1}{\psi_{\max-1}}$  is always  $1/2$  for each  $\Delta$ . Second, note that  $H_{\max-1} > H_{\max-2}$ , but  $H_{\max-2} \rightarrow H_{\max-1} \rightarrow H_{\max}$  as  $\Delta \rightarrow 0$ , whereas the cumulative depth  $Q_{\max-2}$  [using the same logic as in Proposition 8 of Seppi (1997)] converges to something strictly less than  $H_{\max}$  as  $\Delta \rightarrow 0$ . Thus, in the limit  $\beta_{\max-2} \rightarrow H_{\max}$  while  $\theta_{\max-1}^h \rightarrow \frac{1}{2} H_{\max} + \frac{1}{2} \times [\text{something strictly less than } H_{\max}]$  which, by continuity, implies a violation of Condition 1 at  $p_{\max-2}^\Delta$  when the tick size  $\Delta$  is sufficiently small.

*Proof of Proposition 5.* The proof is divided into two parts. First, we show that if an equilibrium exists with hybrid preferencing when  $F \notin \mathcal{F}^\sigma$  and  $p_{\max} > p_{\min} \geq p_2$ , then the HM and PLM limit order books must coexist. Since the PLM book cannot, from Lemmas 9 and 10, be empty when  $F \notin \mathcal{F}^\sigma$ , could the HM book be empty? To see why not, notice that if the HM book is empty, then the specialist executes all HM market orders at  $p_{\max-1}$ . Thus all buying  $x - Q_{\max-2}^p > 0$  is done, given HM preferencing, at  $p_{\max-1}$  in the hybrid market. If  $p_{\max-2} \geq p_{\min}$  so that  $S_{\max-2}^p > 0$ , this implies  $e_{\max-1}^h > e_{\max-2}^p = 0$  if  $S_{\max-1}^h = 0$ , which is a contradiction. If instead  $p_{\max-1} = p_{\min}$ , then, given an empty HM book, all buying would be done from the specialist at  $p_{\max-1}$ . However,  $p_{\max-1} = p_{\min}$  together with  $\theta_{\max-1}^h = 0$  implies  $e_{\max-1}^h > 0$ , which is again a contradiction. Thus the two markets must coexist if an equilibrium with hybrid preferencing exists given  $F \notin \mathcal{F}^\sigma$  and  $p_{\max} > p_{\min} \geq p_2$ .



Second, to prove the existence of an equilibrium given HM preferencing and  $F \notin \mathcal{F}^\sigma$  we use a recursive argument with a function  $\Lambda^{out} = W(\Lambda^{in})$  which we define as follows:

- Given incoming PLM and HM limit orders,  $\Lambda^{in} = (S_1^{p,in}, S_2^{p,in}, \dots, S_1^{h,in}, S_2^{h,in}, \dots)$ , calculate the HM and PLM cost schedules  $T^{h,in}$  and  $T^{p,in}$  induced by these starting limit orders and the specialist's optimal order execution strategy.
- Take the volume distribution  $F$  and the active trader's optimization problem when facing  $T^{h,in}$  and  $T^{p,in}$  and solve for the order arrival distributions  $F^{h,in}$  and  $F^{p,in}$  induced by  $\Lambda^{in}$ . Note that  $F^{h,in}$  may have mass points at market orders equaling the execution thresholds  $\theta_j^{h,in}$  and that  $F^{p,in}$  may have mass points at the cumulative PLM depths  $Q_j^{p,in}$ .
- As part of the definition of the function  $W$ , we create a modified continuous version  $F^{h,mod}$  of the HM order arrival distribution  $F^{h,in}$  by redistributing the probability mass at each of the  $\theta_j^{h,in}$  as a uniform density over the corresponding interval  $[0, \theta_2^{h,in}]$  or  $[\theta_{j-1}^{h,in}, \theta_j^{h,in}]$  immediately below that  $\theta_j^{h,in}$ .
- Similarly we create a continuous version  $F^{p,mod}$  of the PLM order arrival distribution by redistributing any probability mass at each of the  $Q_j^{p,in}$  as a uniform density over the intervals  $[Q_j^{p,in}, Q_{j+1}^{p,in}]$ , where  $Q_{j+1}^{p,in}$  is either the cumulative depth at the next price with positive depth above price  $p_j$  (if any) or else  $\infty$ .
- Since the modified distributions are (by construction) continuous, we can always find unique break-even "outgoing" limit order depths  $\Lambda^{out}$  holding the distributions  $F^{h,mod}$  and  $F^{p,mod}$  fixed.

Note that  $W$  is continuous in  $\Lambda^{in}$  since  $\tau(X)$  is simply a collection of cost line segments corresponding to different cost-minimizing combinations  $(B^h, B^p)$  where the "cross-over" points are continuous in  $\Lambda^{in}$ . In addition,  $W$  is "onto" since the total limit order depths at each price are bounded by 0 and  $H_{max}$ . Thus, the Brouwer fixed point theorem guarantees that a fixed point  $\Lambda = W(\Lambda)$  exists. Finally, note that the  $F^h$  and  $F^p$  corresponding to this fixed point,  $\Lambda$ , have break-even depths without smoothing. To see this, note that the relocation of the probability mass points does not affect the execution probabilities of HM and PLM limit orders at the fixed point  $\Lambda = (S_1^p, S_2^p, \dots, S_1^h, S_2^h, \dots)$  since  $F^{h,in}(B^h > \theta_j^h) = F^{h,mod}(B^h > \theta_j^h)$  and  $F^{p,in}(B^p \geq Q_j^p) = F^{p,mod}(B^p \geq Q_j^p)$ .

*Proof of Proposition 6.* Given that  $\psi_j > 1$  for  $j \geq 2$ , we have at  $p_2$  that  $Q_2^{SHM} = \gamma_2 H_2 + (1 - \gamma_2) H_1 \geq \frac{\gamma_2}{\psi_2} H_2 + (1 - \frac{\gamma_2}{\psi_2}) H_1 = Q_2^{DHM}$ , where the inequality is strict if  $S_2^{SHM} > 0$  and thus  $H_2 > 0$ . Now by induction, if  $Q_{j-1}^{SHM} \geq Q_{j-1}^{DHM}$ , then  $Q_j^{SHM} = \gamma_j H_j + (1 - \gamma_j) Q_{j-1}^{SHM} \geq \frac{\gamma_j}{\psi_j} H_j + (1 - \frac{\gamma_j}{\psi_j}) Q_{j-1}^{DHM} = Q_j^{DHM}$ , where the inequality is again strict if  $S_j^{SHM} > 0$  and thus  $H_j > 0$ .

*Proof of Proposition 7.* Even if the active trader can use PLM limit sells at  $p_j$  to keep the specialist's cleanup price at  $p_{j-1}$ , once  $x > \theta_j^h + Q_j^p$  the active trader has no choice, given global price priority, but to increase her hybrid order to  $B^h > \theta_j^h$ . Once she does this, however, she then maximizes (given HM preferencing) her HM order (up to the next threshold  $\theta_{j+1}^h$ ) and thus initially reduces her PLM order so that  $B^p < Q_j^p$ . Since the set of volumes where the marginal PLM limit order at  $p_j$  executes, but not the marginal HM order (i.e.,  $x = \theta_j^h + Q_j^p$ ), is of measure zero (i.e., given that  $F$  is continuous) and since the set of volumes where (because of HM preferencing) the marginal HM limit sell at  $p_j$  executes but the marginal PLM does not (i.e.,  $\theta_j^h + Q_j^p < x \leq \theta_{j+1}^h + Q_j^p$ ) has positive measure, it follows that  $e_j^p < e_j^h = 0$  and, hence,  $S_j^p = 0$ . Since the PLM is unable to function as a pressure valve in any way (i.e., the PLM book is empty and trade with the PLM crowd at  $p_{max}$  is prohibited by global price priority when  $p^h < p_{max-1}$ ), the only equilibrium with HM preferencing and global price priority is the SHM equilibrium.

*Proof of Proposition 8.* Since a DPLM equilibrium clearly satisfies global price priority given PLM preferencing, consider case (b). Given global price priority, the marginal PLM limit sell at  $p_j$  executes when  $x = \theta_j^h + Q_j^p$  and thus  $B^h = \theta_j^h$  (and  $p^h = p_{j-1}$ ) in the hybrid market and  $B^p = Q_j^p$  (and  $p^p = p_j$ ) in the PLM. However, for any  $x > \theta_j^h + Q_j^p$  global price priority and PLM

preferencing force  $B^h > \theta_j^h$  and  $B^p \geq Q_j^p$ . However,  $\Pr(x \geq \theta_j^h + Q_j^p) = \Pr(x > \theta_j^h + Q_j^p)$  when  $F$  is continuous. Thus  $\Pr(B^h > \theta_j^h) = \Pr(B^p \geq Q_j^p) = \Pr(x \geq \theta_j^h + Q_j^p)$ . Inverting the resulting break-even equilibrium condition  $\Pr(x \geq \theta_j^h + Q_j^p) = \frac{c/\alpha}{p_j - v}$  gives Equation (35). The second condition [Inequality (36)] simply ensures that the equilibrium HM depths  $S_j^h$  lead to strictly ordered HM thresholds  $\theta_j^h < \theta_{j+1}^h$  at prices above  $p_{\min}$  as required by global price priority.

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