# Auctions with Resale Markets: An Exploratory Model of Treasury Bill Markets

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***This article develops a model of competitive bidding with a resale market. The primary market is modeled as a common-value auction, in which bidders participate for the purpose of resale. After the auction the winning bidders sell the objects in a secondary market, and the buyers in the secondary market receive information about the bids submitted in the auction. The effect of this information linkage between the primary auction and the secondary market on bidding behavior in the primary auction is examined. The auctioneer's expected revenues from organizing the primary market as a discriminatory auction versus a uniform-price auction are compared, and sufficient conditions under which the uniform-price auction will yield higher expected revenues are obtained. An example of our model, with the primary market organized as a discriminatory auction, is the U.S. Treasury bill market.***

This article develops an exploratory model for Treasury bill auctions. In these auctions, there are two types of bidders. There are approximately forty competitive bid-

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ders, mainly primary dealers and large financial institutions, who submit sealed bids that are price–quantity pairs. The noncompetitive bidders, primarily individual investors, submit sealed bids that specify quantity only (less than a prespecified maximum). The noncompetitive bids, usually small in quantity,[1] always win. The competitive bidders compete for the remaining bills in a discriminatory auction. That is, the demands of the bidders, starting with the highest price bidder down, are met until all the bills are allocated. The winning competitive bidders pay the unit price they submitted. All the noncompetitive bidders pay the quantity-weighted average price of all the winning competitive bids. After the auction, the Department of the Treasury announces some summary statistics about the bids submitted. These include the total tender amount received, the total tender amount accepted, the highest winning bid, the lowest winning bid, the proportion of winning bids accepted at the lowest price, the quantity-weighted average of winning bids, and the split between competitive and noncompetitive bids. The Treasury bills are then delivered to the winning bidders and can be resold at an active secondary market.

Competitive bidders are large institutions, who tend to have information about the term structure of interest rates that is better than the information possessed by investors in the secondary markets. The buyers on the resale market have access to public information, including information revealed by the Department of the Treasury, about the bids submitted in the auction. To the extent that bids submitted reveal the private information of the competitive bidders, the resale price in the secondary market will be responsive to the bids. This creates an incentive for the primary dealers to signal their private information to the secondary market participants. This information linkage between the actions taken by the competitive bidders in the auction and the resale price is absent in existing models of common-value auctions [see, for example, Milgrom and Weber (1982a)] and is our primary focus.[2]

The revenue-maximizing mechanism for selling Treasury bills was a subject of debate in the early 1960s. Friedman (1960) proposed that the Department of the Treasury should switch from a discriminatory auction to a uniform-price auction for the sale of Treasury bills. Apart from the fact that uniform-price auctions would induce bidders to reveal their true demand curves, Friedman asserted that discriminatory auctions encouraged collusion and discouraged smaller bidders from participating. Both Goldstein (1960) and Brimmer (1962) disputed Friedman's contention. Smith (1966), on the basis of a mathematical model, concluded that uniform-price auctions yield greater revenues. Unlike Smith's model, our model is game-theoretic, in that each bidder's beliefs about the others'

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[1] In the first half of 1989, with and inverted yield curve, the noncompetitive bids increased significantly. For example, in the June 26 auction, noncompetitive bids accounted for about 4 percent of the total amount tendered and about one-sixth of the total amount accepted.

[2] For an analysis of bidding with a resale market when the valuations of the bidders are common knowledge, see Milgrom (1987).

bids are confirmed in an equilibrium and we model the information linkage between the primary auction and the secondary market. Like Smith, our analysis provides support for Friedman's proposal that the Treasury bill auction should be uniform-price.

The results may also be helpful in analyzing other types of auctions with resale markets, such as art auctions, in which bidders have correlated values. If a painting by Van Gogh is auctioned at a price much higher than expected, then one might expect this and other paintings by Van Gogh to be sold at higher prices in the future. In addition, many other financial transactions, although not explicitly conducted as auctions, can nevertheless be thought of as implicitly carried out through auctions; see, for example, an analysis of sales of seasoned new issues in Parsons and Raviv (1985) and the market for corporate control in Tiemann (1988). One feature often shared by these financial transactions is that there exist active resale or secondary markets for the objects for sale. Our analysis here may be helpful in understanding such markets.

The article is organized as follows. In Section 1, we develop a model of competitive bidding with a resale market. The competitive bidders are risk-neutral and have private information about the true value of the objects. After the auction the auctioneer publicly announces some information (the prices paid by the winning bidders, for example) about the auction, and the winning bidders then sell the objects in the secondary market. Although the primary motivation of this model is the Treasury bill market, there are many institutional details which are absent. For instance, we require that competitive bidders demand at most one unit of the object instead of being allowed to choose quantity. And we do not model the effect of any forward contracts and close substitutes (such as last week's Treasury bills) owned by competitive bidders on their bidding strategies. In this article we focus on one aspect of the Treasury bill market—the informational linkage of the resale market and the primary auction.

In Section 2 we analyze discriminatory auctions in which the winning bids and the highest losing bids are revealed at the end of the auction. The equilibrium bids we obtain are higher than those derived in Milgrom and Weber (1982a), because primary bidders have an incentive to signal.

A key insight gained from the theory of common-valuation auctions without resale markets is that the greater the amount of information revealed in an auction, the greater the expected revenue for the auctioneer, since this weakens the "winners' curse." Thus, loosely speaking, if the auctioneer has private information, he can increase his expected revenue by announcing his private information before the auction. However, when there is a resale market that creates an incentive for the bidders to signal, the auctioneer may reduce the bidders' incentive to signal and thus decrease his expected revenue if he announces his private information. Sufficient conditions for the public announcement of the auctioneer's private information to increase his expected revenue are provided.

In Sections 3 and 4 we consider a uniform-price auction in which the rule for determining the winning bidders is identical to the one in the discriminatory auction but the winning bidders pay a uniform price equal to the highest losing bid. The existence of an equilibrium depends in part on the kind of information about the auction publicly revealed by the auctioneer. If, as we assumed for discriminatory auctions, the winning bids are announced, then we show by example that there may exist an incentive for the bidders to submit arbitrarily large bids in order to deceive the secondary market buyers. Bidders in a discriminatory auction do not have such an incentive since, upon winning, they must pay what they bid. However, a symmetric Nash equilibrium always exists in the uniform-price auction, provided that only the price paid by the winning bidders is announced (and thus bidders have no incentive to signal).

Next, we turn to the question of the auctioneer's revenues. The key insight from the theory of auctions without resale markets mentioned above is also useful here. Without resale markets, uniform-price auctions yield higher revenues than discriminatory auctions, since in the former the price is linked to the information of the highest losing bidder. When there are resale markets in which the buyers draw inferences about the true value from the bids and the winning bids *are* announced, there is an additional factor that works in the same direction: namely, that in a uniform-price auction it is cheaper to bid high in order to signal a high realization of private information, since, conditional upon winning, a bidder does not pay what he bids. Therefore, in our model as well, uniform-price auctions result in greater expected revenues for the auctioneer when there exists an equilibrium.

We also provide plausible sufficient conditions for the auctioneer's revenues, under a uniform-price auction in which only the price paid by winning bidders is announced, to be greater than under a discriminatory auction in which all the winning bids are announced. Section 5 contains concluding remarks. All proofs are in the Appendix.

## 1. The Model

Consider a common-value auction in which $n$ risk-neutral bidders (the dealers who submit competitive bids in Treasury bill auctions) bid for $k$ identical, indivisible objects, with $n > k$. The true value of the objects is the same for all the bidders, and is unknown to them at the time they submit bids. Each bidder privately observes a signal about the true value, on the basis of which he submits a bid. We assume that there are no noncompetitive bids. In Section 5 we indicate how noncompetitive bids can be incorporated in our model. Throughout, we assume that each bidder demands (or is allowed) at most one unit of the object.

The competitive bidders' interest in the objects being auctioned is solely for the purpose of resale in the secondary market. We assume that the competitive bidders' personal (consumption) valuations of the object are always sufficiently lower than the valuations of the resale market buyers that they would rather resell the objects than consume them. For instance, the primary dealers may have capital constraints that prevent them from participating in next week's auction unless they sell this week's Treasury bills. Since, as we will establish, competitive bidders make positive expected profits in the auction, they might prefer to sell the Treasury bills at their expected value conditional on all publicly known information.[3]

After the auction, the auctioneer publicly announces some information about the auction. For simplicity we assume that the winning bids and the highest losing bid submitted in the auction are publicly announced. In the case of the uniform-price auction, there may not exist an equilibrium if the winning bids are revealed at the end of the auction. Therefore, we also analyze uniform-price auctions when the winning bids are not announced and when only the price paid by the winning bidders is announced.

We allow the possibility that some additional information about the value of the objects for sale may become publicly available after the bids are submitted but before the opening of the secondary market. The $k$ winners in the primary auction then sell the objects to risk-neutral buyers on the secondary markets. The buyers on the secondary markets do not have access to any private information about the true value. Thus, regardless of the secondary market mechanism—an auction or a competitive market—the resale price will be the expected value of the object conditional on all publicly available information.

The $n$ risk-neutral bidders will be indexed by $i = 1, 2, \ldots, n$. The true value of each object being auctioned is a random variable $\tilde{V}$. Each bidder $i$ has a common prior distribution on $\tilde{V}$ and observes a private signal $\tilde{X}_i$ about the true value. Let $\tilde{P}$ denote any other information that becomes public after the auction is over but before the resale market meets. We will assume, except when otherwise stated, that given $\tilde{P}$ the bidders' signals are not uninformative about the true value, that is, $E(\tilde{V} \mid \tilde{P}) \neq E(\tilde{V} \mid \tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n, \tilde{P})$. If this condition is violated (for example, when $\tilde{P} \equiv \tilde{V}$) our model reduces to the usual common-value auction without a resale market.

Let $f(v, p, \mathbf{x})$ denote the joint density function of $\tilde{V}$, $\tilde{P}$, and the vector of signals $\tilde{X} \equiv (\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n)$. It is assumed that $f$ is symmetric in the last $n$ arguments. Let $[\underline{v}, \bar{v}] \times [\underline{p}, \bar{p}] \times [\underline{x}, \bar{x}]^n$ be the support of $f$, where $[\underline{x}, \bar{x}]^n$ denotes the $n$-fold product of $[\underline{x}, \bar{x}]$. We do not rule out the possibility that the support of the random variables is unbounded either from above or from below. Further, it is assumed that all the random variables in this model are *affiliated*. That is, for all $\mathbf{x}, \mathbf{x}' \in [\underline{x}, \bar{x}]^n$, $v, v' \in [\underline{v}, \bar{v}]$, and $p, p' \in [\underline{p}, \bar{p}]$,

$$f((v, p, \mathbf{x}) \vee (v', p', \mathbf{x}'))f((v, p, \mathbf{x}) \wedge (v', p', \mathbf{x}')) \geq f(v, p, \mathbf{x})f(v', p', \mathbf{x}')$$

[3] If the competitive bidders' personal valuations were the same as that of the resale buyers', and they had superior information, then the resale market would break down.

where $\vee$ denotes the componentwise maximum and $\wedge$ denotes the componentwise minimum. Affiliation is said to be strict if the above inequality is strict. Affiliation implies that if $H$ is an increasing[4] function, then $E[H(\tilde{V}, \tilde{P}, \tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n) \mid c_i \leq \tilde{X}_i \leq d_i, i = 1, \ldots, n]$ is an increasing function of $c_i$, $d_i$. The reader is referred to Milgrom and Weber (1982a) for other implications of affiliation. We further assume for simplicity that if $H$ is continuously differentiable, then $E[H(\tilde{V}, \tilde{P}, \tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n) \mid c_i \leq \tilde{X}_i \leq d_i, i \leq n]$ is continuously differentiable in $c_i$ and $d_i$, for all $c_i, d_i \in [\underline{x}, \bar{x}]$, with the convention that the derivative at $\underline{x}$ is the right-hand derivative and at $\bar{x}$ is the left-hand derivative. Moreover, we will assume that $(\tilde{V}, \tilde{P}, \tilde{X}_1, \ldots, \tilde{X}_n)$ are strictly affiliated, so that if $H$ is strictly increasing in any of $(\tilde{V}, \tilde{P}, \tilde{X}_1, \ldots, \tilde{X}_n)$, say in $\tilde{X}_1$, then $E[H(\tilde{V}, \tilde{P}, \tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n) \mid c_i \leq \tilde{X}_i \leq d_i, i \neq 1]$ is strictly increasing in $c_i$ and $d_i$, for all $c_i, d_i \in [\underline{x}, \bar{x}]$.

## 2. Discriminatory Auction

In a discriminatory auction the bidders submit sealed bids and the $k$ highest bidders win the auction. A winning bidder pays the price he bids. In this section we show that when the competitive bidders' private signals are *information complements,*[5] in a sense to be defined later, there exists a symmetric Nash equilibrium with strategies strictly increasing in bidders' private information. Unlike the auctions examined in Milgrom and Weber (1982a), the affiliation property alone is not sufficient for the existence of a Nash equilibrium. Intuitively, when the motive of the competitive bidders is to resell in a secondary market in which the buyers have information about the bids submitted, there exists an incentive for the competitive bidders to bid more than they otherwise would and thus signal their private information. If each bidder's incentive to signal increases with his information realization, then there exists an equilibrium in strictly increasing strategies. It is the information complementarity of the bidders' signals with respect to the true value that enables the bidders to sort themselves in a separating equilibrium and ensures them a positive expected profit. We also show that if secondary markets participants' beliefs are *monotone,* in a sense to be defined, then there exists a unique symmetric equilibrium.

In a model of an auction of objects for final consumption, Milgrom and Weber (1982a) showed that the auctioneer's expected revenue can be increased if he precommits to truthfully reporting, before the auction, his private information about the objects for sale, provided that his private information is affiliated with the bidders' private information, since this weakens the winners' curse. However, in our model with a resale market

[4] Throughout this article, we will use weak relations. For example, increasing means nondecreasing, positive means nonnegative, etc. If a relation is strict, we will say, for example, strictly increasing.

[5] The reader will see that our notion of information complementarity is different from that in Milgrom and Weber (1982b).

this result is not necessarily true. A portion of the bid submitted by a bidder is attributed to his incentive to signal to the resale market participants. If the auctioneer's private information is a "substitute" for the bidders' information, announcing that information will reduce the responsiveness of the resale price to the bidders' information. This in turn reduces the incentive for the bidders to signal and may cause the expected selling price to fall. On the other hand, when the auctioneer's private information is a "complement" to that of the bidders', it is always beneficial for the auctioneer to announce his private information.

### 2.1 Existence of a symmetric Nash equilibrium

By the hypothesis that $f(v, p, \mathbf{x})$ is symmetric in its last $n$ arguments, this is a symmetric game. Thus, it is natural to investigate the existence of a symmetric Nash equilibrium. We examine the game from bidder 1's point of view. The analysis from the other bidders' viewpoints is symmetric.[6] When bidder 1 submits his bid, he observes only his private information $\tilde{X}_1$, so that a strategy for bidder 1 is a function of $\tilde{X}_1$. We begin our analysis by deriving the first-order necessary conditions for an $n$-tuple $(\hat{b}, \ldots, \hat{b})$ to be a Nash equilibrium in strictly increasing and differentiable strategies, when buyers in the secondary market believe that $(\hat{b}, \ldots, \hat{b})$ are the strategies followed in the bidding.

Since buyers in the secondary market have access only to public information, the resale price is the expectation of $\tilde{V}$ conditional on all public information. As mentioned earlier, we assume that the auctioneer announces the prices paid by winning bidders (i.e., the winning bids) and the highest losing bid.[7] Suppose that bidders $i = 2, \ldots, n$ adopt strategy $\hat{b}$ and that bidder 1 receives information $\tilde{X}_1 = x$ and submits a bid equal to $b$. Then if bidder 1 wins with a bid $b$ and secondary markets buyers believe that he is following $\hat{b}$, the resale price will be

$$\begin{aligned} r^d(\hat{b}^{-1}(b), \tilde{Y}_1, \ldots, \tilde{Y}_k, \tilde{P}) &\equiv E[\tilde{V} | \tilde{X}_1 = \hat{b}^{-1}(b), \hat{b}^{-1}(\hat{b}(\tilde{Y}_1)), \\ &\qquad \ldots, \hat{b}^{-1}(\hat{b}(\tilde{Y}_k)), \tilde{P}] \\ &= E[\tilde{V} | \tilde{X}_1 = \hat{b}^{-1}(b), \tilde{Y}_1, \ldots, \tilde{Y}_k, \tilde{P}] \qquad (1) \end{aligned}$$

where $\hat{b}^{-1}$ denotes the inverse[8] of $\hat{b}$ and $\tilde{Y}_j$ is the $j$th-order statistic of $(\tilde{X}_2, \ldots, \tilde{X}_n)$. Define

$$v^d(x', x, y) \equiv E[r^d(x', \tilde{Y}_1, \ldots, \tilde{Y}_k, \tilde{P}) | \tilde{X}_1 = x, \tilde{Y}_k = y] \qquad (2)$$

[6] To simplify the analysis we arbitrarily assume throughout that in case of a tie, the winner is not chosen randomly. Rather, bidder 1 is declared the winner. This assumption is inconsequential. The equilibrium strategy will remain unchanged if we assume that in case of a tie, the winner(s) is (are) chosen from the tied bidders at random.

[7] If some of the other losing bids, or some function of them, are also announced, all the results remain unchanged.

[8] If $b < \hat{b}(\underline{x})$, then we define $\hat{b}^{-1}(b) = \underline{x}$, and if $b > \hat{b}(\bar{x})$, then $\hat{b}^{-1}(b) = \bar{x}$. Thus, we need to consider only values of $b$ that lie in the range of $\hat{b}$.

which is the expected resale price conditional on $\tilde{X}_1$ and $\tilde{Y}_k$ when bidder 1 wins and the secondary market buyers believe that his private signal is equal to $x'$. By our hypothesis about strict affiliation, both $r^d$ and $v^d$ are strictly increasing in each of their arguments. The expected profit for bidder 1 when $\tilde{X}_1 = x$ and he submits a bid equal to $b$ is

$$\begin{aligned}\pi^d(b|x) &\equiv E[(r^d(\hat{b}^{-1}(b), \tilde{Y}_1, \ldots, \tilde{Y}_k, \tilde{P}) - b)1_{\{b \geq \hat{b}(\tilde{Y}_k)\}} | X_1 = x] \\ &= E[E[(r^d(\hat{b}^{-1}(b), \tilde{Y}_1, \ldots, \tilde{Y}_k, \tilde{P}) - b)1_{\{b \geq \hat{b}(\tilde{Y}_k)\}} | \tilde{X}_1, \tilde{Y}_k] | \tilde{X}_1 = x] \\ &= E[(v^d(\hat{b}^{-1}(b), \tilde{X}_1, \tilde{Y}_k) - b)1_{\{b \geq \hat{b}(\tilde{Y}_k)\}} | \tilde{X}_1 = x] \\ &= \int_{\underline{x}}^{\hat{b}^{-1}(b)} [v^d(\hat{b}^{-1}(b), x, y) - b] f_k(y|x)\, dy \end{aligned} \tag{3}$$

where the first equality follows from the law of iterative expectations and $f_k(y|x)$ denotes the conditional density function of $\tilde{Y}_k$ given $\tilde{X}_1$. For $(\hat{b}, \ldots, \hat{b})$ to be a Nash equilibrium, it is necessary that $\hat{b}$ be a best response for bidder 1 when bidders $i = 2, \ldots, n$ adopt strategy $\hat{b}$ and the resale market participants believe that all bidders adopt $\hat{b}$. That is, the first-order condition for Equation (3) evaluated a $b = \hat{b}$ must be zero:

$$\begin{aligned}0 = \left.\frac{\partial \Pi^d(b|x)}{\partial b}\right|_{b=\hat{b}(x)} &= [v^d(x, x, x) - \hat{b}(x)] f_k(x|x)[\hat{b}'(x)]^{-1} - F_k(x|x) \\ &\quad + [\hat{b}'(x)]^{-1} \int_{\underline{x}}^{x} v_1^d(x, x, y) f_k(y|x)\, dy \end{aligned} \tag{4}$$

where $\hat{b}'(x)$ is the derivative of $\hat{b}(x)$, $F_k(y|x)$ is the conditional distribution function of $\tilde{Y}_k$ given $\tilde{X}_1$, and $v_1^d$ is the partial derivative of $v^d$ with respect to its first argument. Rearranging Equation (4) gives an ordinary differential equation:

$$\hat{b}'(x) = [v^d(x, x, x) - b(x)] \frac{f_k(x|x)}{F_k(x|x)} + \int_{\underline{x}}^{x} v_1^d(x, x, y) \frac{f_k(y|x)}{F_k(x|x)}\, dy \tag{5}$$

Note that, by the definition of $v^d$ and the law of iterative expectations,

$$v^d(x, x, y) = E[\tilde{V} | \tilde{X}_1 = x, \tilde{Y}_k = y] \tag{6}$$

Besides Equation (5), there are two other necessary conditions that $\hat{b}$ must satisfy: (1) $v^d(x, x, x) \geq \hat{b}(x)$, $\forall x \in [\underline{x}, \bar{x}]$, and (2) $\hat{b}(\underline{x}) = v^d(\underline{x}, \underline{x}, \underline{x})$. Condition (1) follows since the expected profit for bidder 1 has to be positive in equilibrium. Condition (2) follows from (1) and the fact that if $\hat{b}(\underline{x}) < v^d(\underline{x}, \underline{x}, \underline{x})$, then by slightly increasing the bid to $b(\underline{x}) + \epsilon$ when $\tilde{X}_1 = \underline{x}$, expected profit can be raised from zero to some strictly positive amount.

The solution to Equation (5) with the boundary condition $\hat{b}(\underline{x}) = v^d(\underline{x}, \underline{x}, \underline{x})$ is

$$\hat{b}(x) = v^d(x, x, x) - \int_{\underline{x}}^{x} L(u|x)\, dt(u) + \int_{\underline{x}}^{x} \frac{b(u)}{f_k(u|u)}\, dL(u|x) \tag{7}$$

where

$$L(u|x) = \exp\left\{-\int_u^x \frac{f_k(s|s)}{F_k(s|s)}\,ds\right\}$$
$$t(u) = v^d(u, u, u)$$
$$h(u) = \int_{\underline{x}}^{u} v_1^d(u, u, y) f_k(y|u)\,dy \tag{8}$$

Note that $L(u|x)$ and $t(u)$ are increasing functions of $u$ and thus are measures on $[\underline{x}, \bar{x}]$. We will show in what follows that $\hat{b}$ of Equation (7) also satisfies condition (1), maximizes expected profit under the hypothesis that $(\tilde{X}_1, \ldots, \tilde{X}_n, \tilde{P})$ are *information complements* with respect to $\tilde{V}$, and is strictly increasing. Essentially, the information complementarity condition guarantees that the responsiveness of the resale price to the submitted bid, as measured by $v_1^d$, increases with a bidder's information realization. Thus, each bidder's incentive to signal increases with his information realization, and this ensures that $\hat{b}$ of Equation (7) is a separating equilibrium strategy.

***Definition 1.*** *Random variables $(\tilde{Z}_1, \ldots, \tilde{Z}_m)$ are said to be information complements with respect to another random variable $\tilde{T}$ if*

$$\frac{\partial^2 \phi(z_1, \ldots, z_m)}{\partial z_i \partial z_j} \geq 0 \qquad \forall i \neq j, \forall z_1, \ldots, z_m$$

*where*

$$\phi(z_1, \ldots, z_m) \equiv E[\tilde{T} | \tilde{Z}_1 = z_1, \ldots, \tilde{Z}_m = z_m]$$

Thus, the random variables $(\tilde{X}_1, \ldots, \tilde{X}_n, \tilde{P})$ are information complements with respect to $\tilde{V}$ if the marginal contribution of $\tilde{X}_1$ to the conditional expectation of $\tilde{V}$ increases with $\tilde{X}_j$ or $\tilde{P}$. This information complementarity condition is satisfied by a large class of distributions. For example, if $\phi(z_1, \ldots, z_m)$ is linear in the $z_i$'s, then $(\tilde{Z}_1, \ldots, \tilde{Z}_m)$ are information complements. Thus, if $(\tilde{T}, \tilde{Z}_1, \ldots, \tilde{Z}_m)$ are multivariate normally distributed, then $(\tilde{Z}_1, \ldots, \tilde{Z}_m)$ are information complements with respect to $\tilde{T}$. Besides the multivariate normally distributed random variables, there is a large class of distributions with linear conditional expectations. The following is an example.

**Example 1.** Let $\tilde{Z}_i$, $i = 1, \ldots, m$ be independent conditional on $\tilde{T}$ and distributed according to the gamma distribution given $\tilde{T} = t$:

$$g_i(z_i | t) = \begin{cases} \dfrac{(1/t)^\alpha}{\Gamma(\alpha)} z_i^{\alpha-1} e^{-z_i/t} & \text{if } z_i > 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha > 0$, $t > 0$, and $\Gamma$ is the gamma function. Let $1/\tilde{T}$ also be distributed according to the gamma distribution with a density

$$h(1/t) = \begin{cases} \dfrac{\sigma^{\gamma}}{\Gamma(\gamma)}\left(\dfrac{1}{t}\right)^{\gamma-1} e^{-\sigma/t} & \text{if } t > 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\gamma > 0$ and $\sigma > 0$. Using Theorem 1 of Milgrom and Weber (1982a), one verifies that $(\tilde{T}, \tilde{Z}_1, \ldots, \tilde{Z}_m)$ are strictly affiliated. Direct computation yields

$$E(\tilde{T} \mid \tilde{Z}_1, \ldots, \tilde{Z}_m) = \frac{\Sigma_{i=1}^{m} \tilde{Z}_i + \sigma}{m\alpha + \gamma - 1}$$

Thus $(\tilde{Z}_1, \ldots, \tilde{Z}_m)$ are information complements with respect to $\tilde{T}$.

Note that the prior distribution of $\tilde{T}$ in Example 1 is an element of the family of "conjugate distributions" of gamma distribution; see DeGroot (1970, chap. 9). Other distributions with linear conditional expectations can be constructed similarly. Interested readers should consult Ericson (1969) and DeGroot (1970). For an example of random variables that are strict information complements, see Bikhchandani and Huang (1989).

The following lemma is a direct consequence of the definition of information complementarity.

***Lemma 1.*** *Suppose that $(\tilde{X}_1, \ldots, \tilde{X}_n, \tilde{P})$ are information complements with respect to $\tilde{V}$. Then $v_1^d(x, x', y)$ is strictly increasing in $x'$ and $y$, where $v_1^d$ denotes the partial derivative of $v^d$ with respect to its first argument.*

*Proof.* See the Appendix. ∎

The following proposition shows that if $(\tilde{X}_1, \ldots, \tilde{X}_n, \tilde{P})$ are information complements with respect to $\tilde{V}$, then $\hat{b}$ of Equation (7) satisfies condition (1), that is, $v^d(x, x, x) \geq \hat{b}(x)\ \forall\ x \in [\underline{x}, \bar{x}]$.

***Proposition 1.*** *Suppose that $(\tilde{X}_1, \ldots, X_n, \tilde{P})$ are information complements with respect to $\tilde{V}$. Then $v^d(x, x, x) \geq \hat{b}(x)$, $\forall\ x \in [\underline{x}, \bar{x}]$ and the inequality is strict for $x > \underline{x}$, where $\hat{b}$ is defined in Equation (7).*

*Proof.* See the Appendix. ∎

A corollary of Proposition 1 is that $\hat{b}$ is strictly increasing. Thus, our assumption that resale market buyers can invert the primary bids to obtain the bidders' signal realizations is justified.

***Corollary 1.*** *The strategy $\hat{b}$ defined in Equation (7) is strictly increasing.*

*Proof.* See the Appendix. ∎

The main result of this section is

***Theorem 1.*** *The n-tuple* $(\hat{b}, \ldots, \hat{b})$, *with* $\hat{b}$ *as defined in Equation (7), is a Nash equilibrium of the discriminatory auction, provided that* $(\tilde{X}_1, \ldots, \tilde{X}_n, \tilde{P})$ *are information complements with respect to* $\tilde{V}$.

*Proof.* See the Appendix. ∎

When bidders in the discriminatory auction participate only for the purpose of consumption, Milgrom and Weber (1982a) have identified a symmetric Nash equilibrium with a bidding strategy

$$b^d(x) = v^d(x, x, x) - \int_{\underline{x}}^{x} L(u \mid x)\, dt(u) \tag{9}$$

where $L(u \mid x)$ and $t(u)$ are as defined in Equation (8). The bidding strategy for the purpose of resale identified in Theorem 1 is strictly higher than $b^d$ for every $x \in (\underline{x}, \bar{x})$ by an amount equal to

$$\int_{\underline{x}}^{x} \frac{h(u)}{f_k(u \mid u)}\, dL(u \mid x)$$

the magnitude of which depends on $v_1^d$, the responsiveness of the resale value to the submitted bid.[9] It is this informational link between the resale value and the bids submitted by bidders in the discriminatory auction that gives the bidders an incentive to signal. Of course, since the $\hat{b}$ is strictly increasing, the resale buyers can invert the bids announced by the auctioneer to obtain the private information of the bidders and, as in Ortega-Reichart (1968) and Milgrom and Roberts (1982), in equilibrium no one gets deceived.

It is worth emphasizing that the competitive bidders benefit if the resale market buyers are better informed about the true value. If, for instance, $\tilde{P} \equiv \tilde{V}$ or $\tilde{P}$ "contains" all the relevant information in $\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n$, then there would be no signaling incentive for the bidders, and the expected price(s) in the auction would be lower. One might expect the "informativeness" of $\tilde{P}$ to increase with the length of time between the end of the primary auction and the start of the resale market.

Next we establish that the symmetric equilibrium identified above is the unique symmetric equilibrium if secondary market buyers' beliefs satisfy a monotonicity condition. The beliefs of the secondary market buyers about the bidders' private information are said to be monotone if they are increasing in the bids submitted. For instance, if the secondary market buyers believe that bidder 1's private information lies in the interval $[x_1^l(b), x_1^u(b)]$ when bidder 1 bids $b$, then the monotonicity condition on beliefs would imply that $x_1^l(\cdot)$ and $x_1^u(\cdot)$ are increasing functions.[10] Since bidders with

[9] If $E[\tilde{V} \mid \tilde{P}] = E[\tilde{V} \mid \tilde{X}_1, \tilde{Y}_1, \ldots, \tilde{Y}_k]$, there is no signaling motive, and the equilibrium strategy is as specified in Equation (9), since $v_1^d(\cdot) \equiv 0$ and $h(\cdot) \equiv 0$.

[10] In the symmetric equilibrium of Theorem 1, the resale buyers' beliefs are monotone (since $\hat{b}$ is increasing) with $x_1^l(b) = x_1^u(b) = \hat{b}^{-1}(b)$. For $b > \hat{b}(\bar{x})$, $x_1^l(b) = x_1^u(b) = \bar{x}$, and for $b < \hat{b}(\underline{x})$, $x_1^l(b) = x_1^u(b) = \underline{x}$.

higher realizations of their private information would expect higher resale prices, it is natural to expect them to bid more. Therefore, monotonicity of beliefs seems to be a natural restriction to impose. Of course, when bids are in the range of the equilibrium bidding strategies, beliefs are obtained by inverting the bidding strategies.

As shown below, $(\hat{b}, \ldots, \hat{b})$ is the only symmetric equilibrium in increasing strategies when the resale buyers' beliefs are monotone. Since a symmetric equilibrium is a natural focal point in a game with symmetric players, we will use this equilibrium when comparing expected revenues between a discriminatory auction and a uniform-price auction.

**Theorem 2.** *If the secondary market buyers have monotone beliefs, then $(\hat{b}, \hat{b}, \ldots, \hat{b})$, where $\hat{b}$ is as defined in Equation (7), is the unique symmetric equilibrium in increasing strategies.*

*Proof.* See Bikhchandani and Huang (1989, theorem 2). ∎

### 2.2 Public announcement of the auctioneer's information

Suppose that the auctioneer has private information about $\tilde{V}$, represented by a random variable $\tilde{X}_0$. We will consider the impact on the expected selling price of announcing $\tilde{X}_0$ before the auction. Let $\bar{b}(\cdot; x_0)$ be a symmetric equilibrium bidding strategy conditional on $\tilde{X}_0 = x_0$. It is assumed that $\bar{b}(\cdot; x_0)$ is increasing and differentiable in $x$. If bidder 1 bids $b$ and wins, then the resale price in this case will be

$$p^d\, \bar{b}^{-1}(b; x_0), \tilde{Y}_1, \ldots, \tilde{Y}_k, \tilde{P}; x_0) \equiv E[\tilde{V} \mid \tilde{X}_1 = \bar{b}^{-1}(b; x_0), \tilde{Y}_1, \ldots, \tilde{Y}^k, \tilde{P}, \tilde{X}_0 = x_0]$$

where $\bar{b}(\bar{b}^{-1}(b; x_0); x_0) = b$. With

$$w^d(x', x, y; x_0) \equiv E[p^d(x', \tilde{Y}_1, \ldots, \tilde{Y}_k, \tilde{P}; x_0) \mid \tilde{X}_1 = x, \tilde{Y}_k = y, \tilde{X}_0 = x_0]$$

it is straightforward to show that $\bar{b}(x; x_0)$ must satisfy

$$\bar{b}'(x; x_0) = [w^d(x, x, x; x_0) - \bar{b}(x; x_0)] \frac{f_k(x \mid x; x_0)}{F_k(x \mid x; x_0)} + \int_{\underline{x}}^{x} w_1^d(x, x, y; x_0) f_k(y \mid x; x_0)\, dy \tag{10}$$

where $f_k(y \mid x; x_0)$ denotes the conditional density of $\tilde{Y}_k$ given $\tilde{X}_1 = x$ and $\tilde{X}_0 = x_0$. In addition, the boundary condition $\bar{b}(\underline{x}; x_0) = w^d(\underline{x}, \underline{x}, \underline{x}; x_0)$ must be satisfied. The solution to Equation (10) with this boundary condition is

$$\bar{b}(x; x_0) = w^d(x, x, x; x_0) - \int_{\underline{x}}^{x} L(u \mid x; x_0)\, dt(u; x_0) + \int_{\underline{x}}^{x} \frac{h(u; x_0)}{f_k(u \mid u; x_0)}\, dL(u \mid x; x_0) \tag{11}$$

where

$$L(u \mid x; x_0) = \exp\left\{-\int_u^x \frac{f_k(s \mid s; x_0)}{F_k(s \mid s; x_0)}\, ds\right\}$$

$$t(u; x_0) = w^d(u, u, u; x_0)$$

$$h(u; x_0) = \int_x^u w_1^d(u, u, y; x_0) f_k(y \mid u; x_0)\, dy$$

Next, we define conditional information complements.

***Definition 2.*** *Random variables $(\tilde{Z}_1, \ldots, \tilde{Z}_m)$ are said to be information complements conditional on random variable $\tilde{Y}$ with respect to random variable $\tilde{T}$ if*

$$\frac{\partial^2 \phi(z_1, \ldots, z_m, y)}{\partial z_i \partial z_j} \geq 0 \qquad \forall \neq j,\ \forall z_1, \ldots, z_m,\ \forall y$$

*where*

$$\phi(z_1, \ldots, z_m, y) \equiv E(\tilde{T} \mid \tilde{Z}_1 = z_1, \ldots, \tilde{Z}_m = z_m, \tilde{Y} = y)$$

If $(\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n, \tilde{P})$ are information complements conditional on $\tilde{X}_0$ with respect to $\tilde{V}$, then a proof identical to that of Theorem 1 shows that $\bar{b}$ defined in Equation (11) is a symmetric equilibrium strategy. This is stated without proof in the following proposition:

***Proposition 2.*** *The n-tuple $(\bar{b}(\cdot\,; x_0), \ldots, \bar{b}(\cdot\,; x_0))$, with $\bar{b}(\cdot\,; x_0)$ as defined in Equation (11), is a Nash equilibrium of the discriminatory auction when the auctioneer announces $\tilde{X}_0 = x_0$, provided that $(\tilde{X}_1, \ldots, \tilde{X}_n, \tilde{P})$ are information complements conditional on $\tilde{X}_0$ with respect to $\tilde{V}$ and the resale market participants believe that all the bidders follow strategy $\bar{b}(\cdot\,; x_0)$.*

Note that the existence of an equilibrium does not depend on whether $(\tilde{X}_0, \tilde{X}_1, \ldots, \tilde{X}_n, \tilde{P})$ are information complements with respect to $\tilde{V}$. There exists a Nash equilibrium as long as $(\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n, \tilde{P})$ are information complements conditional on $\tilde{X}_0$ with respect to $\tilde{V}$. Our main result in this subsection will be that the expected selling price under the policy of always reporting $\tilde{X}_0$ cannot be lower than that under any other reporting policy provided that $(\tilde{X}_0, \tilde{X}_1, \ldots, \tilde{X}_n, \tilde{P})$ are information complements with respect to $\tilde{V}$.[11] We first show, in the next proposition, that $\bar{b}(x; x_0)$ is an increasing function of $x_0$.

[11] Note that this assumption is stronger than the conditional information complementarity required for the existence of a Nash equilibrium.

***Proposition 3.*** *Suppose that* $(\tilde{V}, \tilde{X}_0, \tilde{X}_1, \ldots, \tilde{X}_n, \tilde{P})$ *are affiliated and that* $(\tilde{X}_0, \tilde{X}_1, \ldots, \tilde{X}_n, \tilde{P})$ *are information complements with respect to* $\tilde{V}$. *Then* $\bar{b}(x; x_0)$ *is an increasing function of* $x_0$.

*Proof.* See Bikhchandani and Huang (1989).

The main result of this section is

***Theorem 3.*** *Suppose that* $(\tilde{V}, \tilde{X}_0, \tilde{X}_1, \ldots, \tilde{X}_n, \tilde{P})$ *are affiliated and that* $(\tilde{X}_0, \tilde{X}_1, \ldots, \tilde{X}_n, \tilde{P})$ *are information complements with respect to* $\tilde{V}$. *A policy of publicly revealing the seller's information cannot lower, and may raise, the expected revenue for the seller in a discriminatory auction.*

*Proof.* See the Appendix. ■

Theorem 3 depends critically on the fact that under its hypothesis $\bar{b}(x; x_0)$ is increasing in $x_0$. When $(\tilde{X}_0, \tilde{X}_1, \ldots, \tilde{X}_n, \tilde{P})$ are not information complements, $\bar{b}(x; x_0)$ may not be an increasing function of $x_0$ because the marginal effect of the bid on the resale value may be reduced and revealing $\tilde{X}_0$ may reduce the bidders' incentive to signal. This in turn may lower the expected revenue for the seller even though $(\tilde{X}_0, \tilde{X}_1, \ldots, \tilde{X}_n, \tilde{P})$ are affiliated.

## 3. Uniform-Price Auction

In a uniform-price auction the bidders submit sealed bids and the $k$ highest bidders win the auction. The price they pay is equal to the $(k + 1)$th highest bid. Initially we obtain a candidate for a symmetric Nash equilibrium under the assumption that after the auction the auctioneer reveals the winning bids and the highest losing bid, that is, the price paid by the winning bidders. If any of the lower bids are also revealed, our results remain unchanged.

We first obtain strategies that satisfy first-order necessary conditions for a symmetric Nash equilibrium. Next we show that when there exists a symmetric equilibrium in the uniform-price auction, the auctioneer's expected revenues at this equilibrium are greater than at the symmetric equilibrium of the discriminatory auction. The intuition behind this result is as follows. From the theory of auctions without resale markets we know that, compared with a discriminatory auction, a greater amount of information is revealed during a uniform-price auction. This weakens the winners' curse in the uniform-price auction and results in greater revenues for the auctioneer. In addition, when the primary auction and the resale markets are informationally linked, as in our model, it is cheaper to submit higher bids in the uniform-price auction in order to signal to the resale market buyers. This in turn further increases the expected revenues from uniform-price auctions.

However, the second of these two factors—the fact that in a uniform-price auction the price paid by a bidder conditional upon winning does not increase as his bid is increased—can result in nonexistence of equilibrium in a uniform-price auction. If the resale price is very responsive to the bids submitted, there may exist an incentive for the bidders to submit arbitrarily large bids and upset any purported equilibrium. In the last part of this section we show by example that this is indeed possible and more generally show that when $k = 1$ and $\tilde{P}$ is a constant there does not exist any Nash equilibrium in strictly increasing pure strategies.

### 3.1 Necessary conditions for a symmetric equilibrium

As in the discriminatory auction, each primary bidder's strategy is a function from $[\underline{x}, \bar{x}]$ to the real line. Suppose that $(b_0, b_0, \ldots, b_0)$ is a Nash equilibrium in strictly increasing and differentiable strategies, when buyers in the secondary market believe that all bidders use $b_0$. If all other bidders use strategy $b_0$, bidder 1 receives information $\tilde{X}_1 = x$ and submits a bid equal to $b$, then if bidder 1 wins the resale price is

$$\begin{aligned} r^u(b_0^{-1}(b), \tilde{Y}_1, \ldots, \tilde{Y}_k, \tilde{P}) &\equiv E[\tilde{V} | \tilde{X}_1 = b_0^{-1}(b), b_0^{-1}(b_0(\tilde{Y}_1)), \ldots, \\ &\qquad b_0^{-1}(b_0(\tilde{Y}_k)), \tilde{P}] \\ &= E[\tilde{V} | \tilde{X}_1 = b_0^{-1}(b), \tilde{Y}_1, \ldots, \tilde{Y}_k, \tilde{P}] \end{aligned} \quad (12)$$

Note that $r^u(\cdot) \equiv r^d(\cdot)$ and is strictly increasing in all its arguments. If bidder 1 wins the auction, the expected resale price conditional on $\tilde{X}_1$ and $\tilde{Y}_k$ is

$$v^u(b_0^{-1}(b), x, y) \equiv E[r^u(b_0^{-1}(b), \tilde{Y}_1, \ldots, \tilde{Y}_k, \tilde{P}) | \tilde{X}_1 = x, \tilde{Y}_k = y] \quad (13)$$

By strict affiliation, $v^u$ is strictly increasing in its arguments. From the definition of $v^d$ it follows that

$$v^u(x', x, y) = v^d(x', x, y) \qquad \forall x', x, y \quad (14)$$

We show below that $b_0$ must be given by

$$b_0(x) = v^u(x, x, x) + \frac{h(x)}{f_k(x|x)} \quad (15)$$

where $h(x)$ is as defined in Equation (8).

Using iterative expectations as in Equation (3), if $X_1 = x$ and bidder 1 bids $b$, his expected profit is

$$\Pi^u(b|x) \equiv \int_{\underline{x}}^{b_0^{-1}(b)} [v^u(b_0^{-1}(b), x, y) - b_0(y)] f_k(y|x)\, dy \quad (16)$$

For $(b_0, \ldots, b_0)$ to be a Nash equilibrium, it is necessary that the first-order condition of Equation (16) be zero. That is,

$$0 = b_0'(x) \frac{\partial \Pi^u(b|x)}{\partial b}\bigg|_{b=b_0(x)} = [v^u(x, x, x) - b_0(x)] f_k(x|x) + \int_{\underline{x}}^{x} v_1^u(x, x, y) f_k(y|x)\, dy \quad (17)$$

where $b'_0(x)$ is the derivative of $b_0(x)$, $v_1^u$ is the partial derivative of $v^u$ with respect to its first argument, and we use the assumption that $b_0$ is strictly increasing. Rearranging terms implies that $b_0(x)$ is as defined in Equation (15).

In a uniform-price auction without resale markets Milgrom and Weber (1982a) showed that the symmetric Nash equilibrium bidding strategy is $b^u(x) = v^u(x, x, x,)$. As in discriminatory auctions, the bidding strategy $b_0$ is strictly higher than $b^u$, for every $x \in (\underline{x}, \bar{x}]$, by an amount that depends on $v_1^u$, the responsiveness of the resale value to the submitted bid.

### 3.2 Revenue comparison with the discriminatory auction

Next we show that when there exists a symmetric equilibrium in the uniform-price auction, it generates strictly greater expected revenues than the symmetric equilibrium of the discriminatory auction.[12]

***Theorem 4.*** *When $(b_0, \ldots, b_0)$ is a symmetric Nash equilibrium in the uniform-price auction, the expected revenues generated at this equilibrium are strictly greater than those at the symmetric equilibrium of the discriminatory auction.*

*Proof.* See the Appendix. ■

### 3.3 Possibility of nonexistence of equilibrium

We will illustrate the possibility that strong signaling incentives on the part of the bidders may lead to nonexistence of a pure-strategy Nash equilibrium when winning bids are announced in a uniform-price auction. First we present an example in which max $(\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n, \tilde{P})$ is a sufficient statistic of $(\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n, \tilde{P})$ for the posterior density of $\tilde{V}$. Although in this example the random variables are only weakly affiliated, it illustrates the difficulties that arise when the resale price is very responsive to the winning bids.

**Example 2.** Suppose that all the bids are announced after a uniform-price auction. The marginal density of $\tilde{V}$ is uniform with support [0, 1]. The random variables $(\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n, \tilde{P})$ are identically distributed, are independent conditional on $\tilde{V}$, and their conditional density is uniform on $[0, \tilde{V}]$. Let $\tilde{Z} = \max\{\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n, \tilde{P}\}$. It is readily confirmed that $E[\tilde{V}|\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n, \tilde{P}] = E[\tilde{V}|\tilde{Z}]$. Clearly, the resale price will be equal to $E[\tilde{V}|\tilde{Z}]$, and

$$E[\tilde{V}|\tilde{Z}] \geq \tilde{Z}, \qquad E[\tilde{V}|\tilde{Z} = 1] = 1 \tag{18}$$

[12] An argument similar to that in Theorem 2 establishes that $b_0$ is the only candidate for a symmetric equilibrium when the resale market buyers have monotone beliefs. This remark also applies to the symmetric equilibrium we will obtain in Section 4.

Let $(b_1, b_2, \ldots, b_n)$ be a candidate Nash equilibrium, with $b_i$ strictly increasing. We limit our attention to weakly undominated strategies and thus $0 \leq b_i(\tilde{X}_i) \leq 1$. Let $\Pi^u(b|x)$ be bidder 1's expected payoff when he bids $b$ and $\tilde{X}_1 = x$. Then it is easily verified that if $x < \bar{x}$, then

$$\Pi^u(b_1(x)\,|\,x) \leq \Pi^u(b_1(\bar{x})\,|\,x) \tag{19}$$

since if bidder 1 bids $b_1(\bar{x})$, he will always win whenever he would have won with a bid of $b_1(x)$ and will pay the same price. In addition he will also win whenever the $k$th highest bid of the others' bid is in $(b_1(x), b_1(\bar{x}))$. Moreover, since $b_1$ is strictly increasing, Equation (18) implies that the resale price, if he bids $b_1(\bar{x})$, is equal to one. This is at least as large as the resale price if he wins with a bid of $b_1(x)$. The inequality in Equation (19) is strict as long as there is a nonzero probability that the $k$th highest bid is in the interval $(b_1(x), b_1(\bar{x}))$.

Thus, the only candidate Nash equilibrium appears to be a somewhat degenerate one in which at least one of the bidders, say bidder 1, always bids one, at least $n - k$ bidders bid sufficiently low that they never win, and the $(k + 1)$th highest bid is low enough that bidder 1 always maximizes his profits by bidding one. But if there is any strictly positive cost of participating in the auction (such as an entry fee or a bid preparation cost), then the $n - k$ bidders who never win at this equilibrium will not participate in the auction.

Next we show that if $k = 1$ and if $\tilde{P}$ is totally uninformative about $\tilde{V}$, then there does not exist a symmetric Nash equilibrium in strictly increasing strategies. Essentially, when there is only one object and no other information becomes public after the auction, there exists a large incentive for the bidders to submit high bids, since the resale price is very responsive to the winning bid.

***Proposition 4.*** *Suppose that $k = 1$ and that $\tilde{P}$ is independent of $\tilde{V}$. Then, if the winning bid and the highest losing bid are announced, there does not exist a Nash equilibrium in strictly increasing strategies.*

*Proof.* See the Appendix. ■

However, there exist other examples for which an equilibrium exists. For instance, if $\tilde{V}$ is uniform on [0, 1] and $\tilde{X}_1$ is uniform on $[\tilde{V}, 1]$, then $v_1^u(\cdot, \cdot, \cdot) \equiv 0$ and the equilibrium strategy is to bid $b_0(x) = v^u(x, x, x)$. Whether there exist intuitive sufficient conditions under which an equilibrium exists remains an open question.

## 4. Uniform-Price Auction without Signaling

Since there exists a possibility of nonexistence of equilibrium in a uniform-price auction with signaling, in this section we analyze uniform-price auctions when the winning bids are not announced. Thus, the bidders do not have an incentive to signal their private information, since if they win their bids are not revealed. Even without the signaling incentive, we are able to show that in at least two scenarios the expected revenue generated by this uniform-price auction is higher than that generated by the discriminatory auction with signaling discussed in Section 2. We believe that one of these two scenarios is plausible for the case of the Treasury bill market.

### 4.1 Existence of a symmetric Nash equilibrium when winning bids are not announced

We show below that there exists a symmetric Nash equilibrium in strictly increasing and differentiable strategies $(b^*, b^*, \ldots, b^*)$ when buyers in the secondary market believe that all bidders use $b^*$. Suppose that bidders $i = 2, \ldots, n$ adopt the strategy $b^*$ and bidder 1 receives information $\tilde{X}_1 = x$ and submits a bid equal to $b$. If bidder 1 wins, the resale price is

$$\begin{aligned}\hat{r}^u(\tilde{Y}_k, \tilde{P}) &\equiv E[\tilde{V} \mid \tilde{Z}_{k+1} = \tilde{Y}_k, \tilde{P}] \\ &= E[\tilde{V} \mid \tilde{X}_1 \geq \tilde{Y}_k, \tilde{Y}_k, \tilde{P}]\end{aligned}$$

where $\tilde{Z}_j$ is the jth-order statistic of $(\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n)$. The equality follows from the fact that the signals are identically distributed. Note that $\hat{r}^u$ is strictly increasing in both its arguments. If bidder 1 wins the auction, the expected resale price conditional on $\tilde{Y}_k$ and $\tilde{X}_1$ is

$$\hat{v}^u(x, y) \equiv E[\hat{r}^u(\tilde{Y}_k, \tilde{P}) \mid \tilde{X}_1 = x, \tilde{Y}_k = y]$$

By strict affiliation, $\hat{v}^u$ is strictly increasing in its arguments. Thus, if $X_1 = x$ and bidder 1 bids $b$, his expected profit is

$$\begin{aligned}\hat{\Pi}^u(b \mid x) &\equiv E[(\hat{r}^u(\tilde{Y}_k, \tilde{P}) - b^*(\tilde{Y}_k))\, 1_{\{b \geq b^*(\tilde{Y}_k)\}} \mid \tilde{X}_1 = x] \\ &= E[E[(\hat{r}^u(\tilde{Y}_k, \tilde{P}) - b^*(\tilde{Y}_k))\, 1_{\{b \geq b^*(\tilde{Y}_k)\}} \mid \tilde{X}_1, \tilde{Y}_k] \mid \tilde{X}_1 = x] \\ &= E[(\hat{v}^u(\tilde{X}_1, \tilde{Y}_k) - b^*(\tilde{Y}_k))\, 1_{\{b \geq b^*(\tilde{Y}_k)\}} \mid \tilde{X}_1 = x]. \qquad (20)\end{aligned}$$

Define

$$b^*(x) \equiv \hat{v}^u(x, x) \qquad (21)$$

Note that $b^*$ is strictly increasing. We show that $(b^*, b^*, \ldots, b^*)$ is an equilibrium.

***Theorem 5.*** *The n-tuple $(b^*, b^* \ldots, b^*)$ is a Nash equilibrium in the uniform-price auction provided that resale market buyers believe that all the bidders follow the strategy $b^*$.*

*Proof.* see the Appendix. ■

Milgrom and Weber (1982a) have shown that the price paid by winning bidders in a uniform-price auction when bidders participate for the purposes of consumption is $E(\tilde{V} \mid \tilde{X}_1 = \tilde{Y}_k, \tilde{Y}_k)$. We show in the next lemma that the price paid by winning bidders in the uniform-price auction in our model is greater than this. This is true even though the competitive bidders do not have a signaling motive.

***Lemma 2.*** *With probability 1,*

$$b^*(\tilde{Y}_k) > E[\tilde{V} \mid \tilde{X}_1 = \tilde{Y}_k, \tilde{Y}_k]$$

*where the right-hand side is the price paid by winning bidders in Milgrom and Weber (1982a) in a uniform-price auction without resale markets (in which the bidders participate for consumption).*

*Proof.* See Bikhchandani and Huang (1989, lemma 2). ■

The "true value" of the object for the competitive bidders is the resale price. Thus, if no additional information becomes available after the auction, that is, if $\tilde{P}$ is constant, the winners' curse on the primary bidders is weakened. Since there is no signaling motive, one would expect the bids in the primary auction to increase when $\tilde{P}$ is constant [or when $\tilde{P}$ is independent of $(\tilde{V}, \tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n)$]. This is proved in the next lemma.

***Lemma 3.*** *With probability 1, the bids in the uniform-price auction strictly increase when $\tilde{P}$ is constant, that is, when no additional information (other than about the bids submitted in the auction) becomes public after the auction.*

*Proof.* See Bikhchandani and Huang (1989, lemma 3). ■

### 4.2 Revenue comparison with the discriminatory auction

We obtain two sets of sufficient conditions under which the expected revenues generated at the symmetric equilibrium of the uniform-price auction, obtained in Section 4.1, are greater than the expected revenues at the symmetric equilibrium of the discriminatory auction of Section 2. The first set seems plausible for the case of Treasury bill auctions.

The following theorem states that if the public information, $\tilde{P}$, is not very informative about the true value of the objects, the uniform-price auction generates higher expected revenue than the discriminatory auction.

***Theorem 6.*** *There exists a scalar $M > 0$ such that if $\partial \hat{r}^u(y, p)/\partial p \leq M$ for all $y, p \in [\underline{x}, \bar{x}] \times [\underline{p}, \bar{p}]$, then the uniform-price auction without signaling generates strictly higher expected revenue than the discriminatory auction with signaling, at their respective symmetric equilibria.*

*Proof.* See the Appendix. ■

Theorem 6.says that if the ex post public information $\tilde{P}$ has little impact on the resale price conditional on the information released from the uniform-price auction, then the auctioneer's expected revenue is higher in the uniform-price auction even when winning bids are not announced.

This is true even though there is no signaling aspect in the uniform-price auction. In the case of the Treasury bill auction, bids are submitted before 1:00 P.M. every Monday. The results of the auction are announced around 4:30 P.M., and the resale market comes into play. One would expect that any public information that normally arrives between 1:00 P.M. and 4:30 P.M. would not be very informative about $\tilde{V}$ conditional on the results of the earlier auction.

The following theorem gives an alternative scenario under which, once again, the uniform-price auction generates higher revenues. Essentially, it says that if the signaling motive of the bidders is not strong, then the uniform-price auction generates higher expected revenue.

***Theorem 7.*** *Suppose that $(\tilde{V}, \tilde{X}_1, \ldots, \tilde{X}_n)$ are strictly affiliated. There exists a scalar $M > 0$ such that if $\partial r^d(x, y_1, \ldots, Y_k, p)/\partial x \leq M$ for all $x, y_1, \ldots, y_k, p \in [\underline{x}, \bar{x}]^{k+1} \times [\underline{p}, \bar{p}]$, then the uniform-price auction without signaling generates strictly higher expected revenue than the discriminatory auction with signaling, at their respective symmetric equilibria.*

*Proof.* See the Appendix. ■

A scenario in which Theorem 7 is applicable is when the public information $\tilde{P}$ is very informative about the true value $\tilde{V}$. Then the impact of $\tilde{X}_1$ on the resale price will be small when bidder 1 wins. This scenario, however, does not seem to be plausible in the case of Treasury bill auctions.

## 5. Conclusions

This article is an exploratory study of competitive bidding when there exists a resale market that is informationally linked to the bidding. We have shown that there exists a symmetric Nash equilibrium in discriminatory auctions when winning bids and the highest losing bid are announced, provided that the relevant variables are affiliated and are information complements. This is the only symmetric equilibrium when the resale market buyers have monotone beliefs. If his information is complementary to that of the bidders, the auctioneer will increase his expected revenue by precommitting to announcing his private information before the auction. We know little about the general impact of the ex post information $\tilde{P}$ on the signaling motive of bidders. For the case of Treasury bill markets this is not important since we believe that very little additional information becomes publicly available in the short time period between the closing of the Treasury bill auction and the opening of the resale market.

We established the possibility of nonexistence of an equilibrium in a uniform-price auction when winning bids are announced. However, when there exists a symmetric equilibrium, it generates greater expected revenues than the symmetric equilibrium in a discriminatory auction. We also showed that there exists a symmetric Nash equilibrium in a uniform-price auction when only the highest losing bid is announced, so that there is no signaling incentive for the bidders. Two scenarios were provided in which the uniform-price auction without signaling generates strictly higher expected revenue for the auctioneer than the discriminatory auction.

Some implications of our model deserve attention. First, it is easy to incorporate noncompetitive bids into our model, provided the total amount of noncompetitive bids $j$ is common knowledge before the auction. We can model the primary auction as one with $n$ bidders and $k + j$ objects, $j$ of which are awarded to noncompetitive bidders at the average price. The preceding analysis remains unchanged. The expected profits of the noncompetitive bidders are positive and equal to the ex ante expected profits of the competitive bidders. There is a prespecified minimum and maximum quantity for each noncompetitive bid, which may be the reason that resale market buyers do not buy Treasury bills solely through noncompetitive bids; and it may also explain why competitive bidders do not submit only noncompetitive bids while avoiding (presumably costly) information collection.[13] Second, the fact that ex ante expected profits of the bidders are strictly positive (except in uniform-price auctions without signaling, when $\tilde{P}$ is uninformative about $\tilde{V}$) implies that in expectation, the average price in the auction is strictly less than the resale price. This comparison has been empirically documented by Cammack (1986). Third, as pointed out in Section 2.1, the competitive bidders benefit if the resale market buyers are better informed about the true value, since it decreases the bidders' signaling incentive.

There are several possible extensions of our model. First, in the Treasury bill auction the competitive bidders submit price–quantity pairs and demand more than one unit of the Treasury bills. This feature is missing in our model. Second, two weeks before the weekly Treasury bill auction, forward contracts of the Treasury bills to be auctioned are traded among the competitive bidders. The relationship among the forward prices, bids submitted, and the resale price needs to be investigated. Third, there exists a wide variety of close substitutes of Treasury bills carried as inventories by competitive bidders. These close substitutes may have a significant effect on the interplay between the forward markets and the weekly auction. We hope to investigate some of these issues in future research.

## Appendix

### Proof of Lemma 1

The joint density of $(\tilde{V}, \tilde{P}, \tilde{X}_1, \tilde{Y}_1, \ldots, \tilde{Y}_{n-1})$ is

$$(n-1)!f(v, p, x, y_1, \ldots, y_{n-1})1_{\{y_1 \geq y_2 \geq \cdots \geq y_{n-1}\}}$$

As a consequence, the conditional density of $\tilde{V}$ given $(\tilde{P}, \tilde{X}_1, \tilde{Y}_1, \ldots, \tilde{Y}_{n-1})$ is

[13] By renormalizing the units, we can assume that each competitive bidder demands exactly $m > 1$ or zero units. This allows us to keep the prespecified maximum demand of noncompetitive bidders at a level less than that of competitive bidders.

$$\frac{f(v, p, x, y_1, \ldots, y_{n-1})}{f(p, x, y_1, \ldots, y_{n-1})} 1_{\{y_1 \geq y_2 \geq \cdots \geq y_{n-1}\}}$$

Thus $(\tilde{P}, \tilde{X}_1, \tilde{Y}_1, \ldots, \tilde{Y}_k)$ are information complements. Let $r_1^d$ denote the derivative of $r^d$, which is defined in Equation (1), with respect to its first argument. It is then easily verified that $r_1^d(x, y_1, \ldots, y_k, p)$ is a strictly increasing function of $p$, $y_j$, $\forall\ j$. Next note that

$$v_1^d(x, x', y) = E[r_1^d(x, \tilde{Y}_1, \ldots, \tilde{Y}_k, \tilde{P}) \mid \tilde{X}_1 = x', \tilde{Y}_k = y]$$

Theorem 5 of Milgrom and Weber (1982a) then implies, by affiliation, that $v_1^d(x, x', y)$ is a strictly increasing function of $x'$ and $y$. ■

**Proof of Proposition 1**

We first write

$$v^d(x, x, x) - \hat{b}(x) = \int_{\underline{x}}^{x} L(u|x)\ dt(u) - \int_{\underline{x}}^{x} \frac{h(u)}{f_k(u|u)}\ dL(u|x)$$

$$= \int_{\underline{x}}^{x} L(u|x)\left[ v_1^d(u, u, u) + v_2^d(u, u, u) + v_3^d(u, u, u) - \frac{h(u)}{F_k(u|u)}\right] du \quad \text{(A1)}$$

By the hypothesis that $(\tilde{X}_1, \ldots, \tilde{X}_n, \tilde{P})$ are information complements with respect to $\tilde{V}$ and Lemma 1, $v_1^d(u, u, u) \geq v_1^d(u, u, y)$ for $y \leq u$. It follows that

$$\frac{h(u)}{F_k(u|u)} = \int_{\underline{x}}^{u} v_1^d(u, u, y) \frac{f_k(y|u)}{F_k(u|u)}\ dy \leq v_1^d(u, u, u)$$

Substituting this relation into Equation (A1) gives

$$v^d(x, x, x) - \hat{b}(x) \geq \int_{\underline{x}}^{x} L(u|x)[v_2^d(u, u, u) + v_3^d(u, u, u)]\ du \geq 0$$

Note that the above inequality is strict for $x \in (\underline{x}, \bar{x}]$ since $v_2^d > 0$ and $v_3^d > 0$ by strict affiliation. ■

**Proof of Corollary 1**

We will show that $\hat{b}'(x) > 0\ \forall x > \underline{x}$. From Proposition 1 we have $v^d(x, x, x) - \hat{b}(x) > 0\ \forall\ x > \underline{x}$. The proof is completed by inserting this in Equation (5) and noting that $v_1^d > 0$. ■

**Proof of Theorem 1**

Let $x' \leq x$. Recall from Equation (4) that

$$
\begin{aligned}
0 = \frac{\partial \Pi^d(\hat{b}(x')\,|\,x')}{\partial b} &= [\hat{b}'(x')]^{-1}F_k(x'|\,x') \\
&\quad \cdot \Bigg\{ [v^d(x', x', x') - \hat{b}(x')]\frac{f_k(x'|\,x')}{F_k(x'|\,x')} - \hat{b}'(x') \\
&\quad + \int_{\underline{x}}^{x'} v_1^d(x', x', y)\,\frac{f_k(y|\,x')}{F_k(x'|\,x')}\,dy \Bigg\} \\
\leq [\hat{b}'(x')]^{-1}F_k(x'|\,x') &\Bigg\{ [v^d(x', x, x') - \hat{b}(x')]\frac{f_k(x'|\,x)}{F_k(x'|\,x)} - \hat{b}'(x') \\
&\quad + \int_{\underline{x}}^{x'} v_1^d(x', x, y)\,\frac{f_k(y|\,x')}{F_k(x'|\,x')}\,dy \Bigg\} \\
\leq [\hat{b}'(x')]^{-1}F_k(x'|\,x') &\Bigg\{ [v^d(x', x, x') - \hat{b}(x')]\frac{f_k(x'|\,x)}{F_k(x'|\,x)} - \hat{b}'(x') \\
&\quad + \int_{\underline{x}}^{x'} v_1^d(x', x, y)\,\frac{f_k(y|\,x)}{F_k(x'|\,x)}\,dy \Bigg\} \\
= \left[\frac{F_k(x'|\,x')}{F_k(x'|\,x)}\right] &\left[\frac{\partial \Pi^d(\hat{b}(x')\,|\,x)}{\partial b}\right]
\end{aligned}
$$

where the first inequality follows from Proposition 1, Lemma 1, and the fact that $F_k(y|\,x)/f_k(y|\,x)$ is decreasing in $x$ [see Milgrom and Weber (1982a)], and the second inequality follows from the fact that if $x' \geq x \geq y$, then the distribution function $F_k(\cdot\,|x')/F_k(x'|\,x')$ dominates the distribution function $F_k(\cdot\,|x)/F_k(x|\,x)$ in the sense of first-order stochastic dominance [see lemma 3 of Bikhchandani and Huang (1989)]. That is, when $\tilde{X}_1 = x$ and bidder 1 bids $b = \hat{b}(x') \leq \hat{b}(x)$, his expected profit can be raised by bidding higher. Similar arguments show that $\partial \Pi^d(b(x')\,|\,x)/\partial b \leq 0$ for $x' \geq x$. As a consequence, $\Pi^d(b|\,x)$ is maximized at $b = \hat{b}(x)$. Finally, since $\Pi^d(\hat{b}(\underline{x})\,|\,x) = 0$ for all $x$, we have $\Pi^d(\hat{b}(x)\,|\,x) > 0$ for all $x > \underline{x}$ by strict affiliation. We have thus shown that $\hat{b}(x)$ is the best strategy for bidder 1 when he observes $\tilde{X}_1 = x$, when bidders $i = 2, 3, \ldots, n$ follow $\hat{b}$, and when the resale market participants believe that all the bidders follow $\hat{b}$. ■

**Proof of Theorem 3**

The proof mimics that of Milgrom and Weber (1982a, theorem 16) and is omitted. The interested reader is referred to Bikhchandani and Huang (1989, theorem 3) for details. ■

**Proof of Theorem 4**

Let $P^u(x)$ denote the expected price paid by bidder 1 in a uniform-price auction, conditional upon winning, when bidders use strategy $b_0$ and $\tilde{X}_1 = x$. That is,

$$P^u(x) \equiv \int_{\underline{x}}^{x} b_0(y) \frac{f_k(y|x)}{F_k(x|x)}\, dy$$
$$= \int_{\underline{x}}^{x} \left[ v^u(y, y, y) + \frac{h(y)}{f_k(y|y)} \right] \frac{f_k(y|x)}{F_k(x|x)}\, dy \tag{A2}$$

If $P^d(x)$ denotes the corresponding expected price when bidders use strategy $\hat{b}$ in a discriminatory auction, then

$$P^d(x) \equiv \hat{b}(x) \tag{A3}$$
$$= \int_{\underline{x}}^{x} \left[ v^d(y, y, y) + \frac{h(y)}{f_k(y|y)} \right] dL(y|x)$$
$$= \int_{\underline{x}}^{x} b_0(y)\; dL(y|x) \tag{A4}$$

where the first equality follows from using integration by parts on Equation (7) and the second from Equation (14). Note that $L(u|x)$ and $F_k(y|x)/F_k(x|x)$ are probability distributions on $[\underline{x}, x]$. Then, since $b_0(y)$ is increasing, the proof is complete if we can show that $F_k(y|x)/F_k(x|x)$ stochastically dominates $L(u|x)$ in the sense of first-order, that is, $F_k(y|x)/F_k(x|x) \le L(u|x)$, $\forall y$.

Since

$$\ln[F_k(y|x)] = \int_{\underline{x}}^{y} \frac{f_k(s|x)}{F_k(s|x)}\, ds$$

we have

$$F_k(y|x) = \exp\left[ \int_{\underline{x}}^{y} \frac{f_k(s|x)}{F_k(s|x)}\, ds \right]$$

Therefore

$$\frac{F_k(y|x)}{F_k(x|x)} = \exp\left[ -\int_{y}^{x} \frac{f_k(s|x)}{F_k(s|x)}\, ds \right]$$
$$\le \exp\left[ -\int_{y}^{x} \frac{f_k(s|s)}{F_k(s|s)}\, ds \right]$$
$$= L(y|x)$$

where the inequality follows from Bikhchandani and Huang (1989, lemma 3). ■

**Proof of Proposition 4**

Let $(b_1, b_2, \ldots, b_n)$ be a candidate for Nash equilibrium, where $b_i\colon [\underline{x}, \bar{x}] \to \Re$ are strictly increasing. We will show that if bidder $j$ uses strategy $b_j$, $j = 2, 3, \ldots, n$ and the resale market buyers believe that each bidder $i$

uses strategy $b_i$, $i = 1, 2, \ldots, n$, then bidder 1 has an incentive to deviate from $b_1(\tilde{X}_1)$. In fact, we will show that when $\tilde{X}_1 = x$, bidder 1's profits are *minimized* at a bid of $b_1(x)$.

The price that bidder 1 faces is

$$\tilde{B}_1 \equiv \max\,[b_2(\tilde{X}_2),\ b_3(\tilde{X}_3),\ \ldots,\ b_n(\tilde{X}_n)]$$

Since $b_i$, $i = 2, 3, \ldots, n$ are strictly increasing, $\tilde{X}_1$ and $\tilde{B}_1$ are strictly affiliated and $\tilde{B}_1$ is atomless. We assume, for simplicity, that $\tilde{B}_1$ has a density function. The expected resale price if bidder 1 wins with a bid equal to $b$ is[14]

$$r^u(b_1^{-1}(b),\ \tilde{B}_1) \equiv E[\tilde{V} | \tilde{X}_1 = b_1^{-1}(b),\ \tilde{B}_1]$$

The expected profit for bidder 1 if $\tilde{X}_1 = x$ and he bids $b_1(x')$ is

$$\Pi^u(x'|x) \equiv E\{[r^u(x', \tilde{B}_1) - B_1]1_{\{b_1(x') \geq \tilde{B}_1\}} | \tilde{X}_1 = x\}$$
$$= \int_{\underline{b}}^{b_1(x')} [r^u(x', \beta) - \beta] g(\beta | x)\, d\beta$$

where $\underline{b} \equiv \min\,[b_2(\underline{x}),\ b_3(\underline{x}),\ \ldots,\ b_n(\underline{x})]$ and $g(\cdot | x)$ is the conditional density of $\tilde{B}_1$ given $\tilde{X}_1 = x$. The first-order necessary condition for $b_1$ to be an equilibrium strategy is

$$\left.\frac{\partial \Pi^u(x'|x)}{\partial x'}\right|_{x'=x} = [r^u(x, b_1(x)) - b_1(x)] g(b_1(x) | x)$$
$$+ \int_{\underline{b}}^{b_1(x)} r_1^u(x, \beta) g(\beta | x)\, d\beta = 0$$

where $r_1^u(x, \beta)$ denotes the derivative of $r^u$ with respect to its first argument.

Let $x' > x$. Then by strict affiliation

$$0 = [r^u(x', b_1(x')) - b_1(x')] g(b_1(x') | x') + \int_{\underline{b}}^{b_1(x')} r_1^u(x', \beta) g(\beta | x')\, d\beta$$
$$< g(b_1(x') | x') \left\{ [r^u(x', b_1(x')) - b_1(x')] + \int_{\underline{b}}^{b_1(x')} r_1^u(x', \beta) \frac{g(\beta | x)}{g(b_1(x') | x)}\, d\beta \right\}$$
$$= \frac{g(b_1(x') | x')}{g(b_1(x') | x)} \frac{\partial \Pi^u(x'|x)}{\partial x'}$$

Similarly, we can show that, for $x' < x$,

[14] Here we assume that the identity of the winning bidder is also disclosed. In our earlier analysis, since we restricted attention to symmetric equilibria, such an assumption was not necessary.

$$\frac{\partial \Pi(x'|x)}{\partial x'} < 0$$

Thus, for any $x \in [\underline{x}, \bar{x}]$, $\Pi^u(x'|x)$ achieves a global minimum at $x' = x$! ■

**Proof of Theorem 5**

Given that bidders 2, 3, . . . , $n$ use $b^*$, we can rewrite Equation (20) as

$$\hat{\Pi}^u(b|x) = \int_{\underline{x}}^{b^{*-1}(b)} [\hat{v}^u(x, y) - \hat{v}^u(y, y)] f_k(y|x)\, dy \tag{A6}$$

where $f_k(y|x)$ is the conditional density of $\tilde{Y}_k$ given $\tilde{X}_1$. Since, by strict affiliation, $\hat{v}^u$ is strictly increasing in both arguments, the integrand in Equation (A6) is positive if and only if $x \geq y$. Thus, bidder 1's profits are maximized when he wins if and only if $\{\tilde{X}_1 \geq \tilde{Y}_k\}$. Therefore, bidder 1's profits are maximized[15] if he uses strategy $b^*$. ■

**Proof of Theorem 6**

By symmetry, the expected revenue for the auctioneer is equal to $n$ times the unconditional expected payment of bidder 1. Let $\hat{R}^u$ and $R^d$ denote the expected revenue of the auctioneer under uniform-price auction and under discriminatory auction, respectively. Then

$$\begin{aligned} \hat{R}^u &= n \times E[b^*(\tilde{Y}_k) 1_{\{\tilde{X}_1 \geq \tilde{Y}_k\}}] \\ &= k \times E[b^*(\tilde{Y}_k) \,|\, \tilde{X}_1 \geq \tilde{Y}_k] \\ R^d &= n \times E[\hat{b}(\tilde{X}_1) 1_{\{\tilde{X}_1 \geq \tilde{Y}_k\}}] \\ &= k \times E[\hat{b}(\tilde{X}_1) \,|\, \tilde{X}_1 \geq \tilde{Y}_k] \end{aligned}$$

Note that the total unconditional expected profits for bidders in equilibrium for the uniform-price auction and for the discriminatory auction are, respectively,

$$n \times E\{[\hat{r}^u(\tilde{Y}_k, \tilde{P}) - b^*(\tilde{Y}_k)] 1_{\{\tilde{X}_1 \geq \tilde{Y}_k\}}\} = k \times E[\tilde{V}|\tilde{X}_1 \geq \tilde{Y}_k] - \hat{R}^u$$

and

$$n \times E\{[r^d(\tilde{X}_1, \tilde{Y}_1, \ldots, \tilde{Y}_k, \tilde{P}) - b^*(\tilde{X}_1)] 1_{\{\tilde{X}_1 \geq \tilde{Y}_k\}}\} = k \times E[\tilde{V}|\tilde{X}_1 \geq \tilde{Y}_k] - R^d$$

Thus, before bidders receive their private information, the two auctions are constant-sum games between the auctioneer and the bidders, with total payoff equal to $k \times E(\tilde{V}|\tilde{X}_1 \geq \tilde{Y}_k)$. Putting

$$\hat{r}^u(\tilde{Y}_k) \equiv E[\tilde{V}|\tilde{X}_1 \geq \tilde{Y}_k, Y_k]$$

it is easily seen that

[15] If $\tilde{P}$ is independent of $\tilde{X}_i$, or if there is no postauction public information, then $\hat{v}^u$ is constant in its first argument and no bid gives bidder $i$ an expected profit greater than zero. However, $(b^*, \ldots, b^*)$ remains an equilibrium.

$$E[\hat{r}^u(\tilde{Y}_k, \tilde{P}) - \hat{r}^u(\tilde{Y}_k) \mid \tilde{X}_1 \geq \tilde{Y}_k, \tilde{Y}_k] = 0$$

and hence

$$E[\hat{r}^u(\tilde{Y}_k, \tilde{P}) - \hat{r}^u(\tilde{Y}_k) \mid \tilde{X}_1 \geq \tilde{Y}_k] = 0$$

Thus

$$R^o \equiv k \times E[\hat{r}^u(\tilde{Y}_k) \mid \tilde{X}_1 \geq \tilde{Y}_k] > R^d = k \times E[\hat{b}(\tilde{X}_1) \mid \tilde{X}_1 \geq \tilde{Y}_k]$$

since the (unconditional) expected profit of a bidder in a discriminatory auction is always strictly positive by strict affiliation.

For ease of exposition, we assume that the support of $\tilde{P}$ is finite. Then $M \equiv (R^o - R^d)/[k(\bar{p} - \underline{p})]$ is strictly positive. We will show that if $\partial \hat{r}^u(y, p)/\partial p \leq M$ for all $y, p \in [\underline{x}, \bar{x}] \times [\underline{p}, \bar{p}]$, then $\hat{R}^u \geq R^d$. First note that by affiliation $\partial \hat{r}^u(y, p)/\partial p \geq 0$. Thus

$$\begin{aligned} \hat{r}^u(\tilde{Y}_k, \tilde{P}) &\geq \hat{r}^u(\tilde{Y}_k, \bar{p}) - M(\bar{p} - \tilde{P}) \\ &> \hat{r}^u(\tilde{Y}_k) - M(\bar{p} - \underline{p}) \end{aligned}$$

where the strict inequality follows from the assumption of strict affiliation. Hence

$$\hat{r}^u(\tilde{Y}_k) - \hat{r}^u(\tilde{Y}_k, \tilde{P}) < \frac{R^o - R^d}{k}$$

and thus

$$\begin{aligned} -b^*(\tilde{Y}_k) &= -E[\hat{r}^u(\tilde{Y}_k, \tilde{P}) \mid \tilde{X}_1 = \tilde{Y}_k, \tilde{Y}_k] \\ &< \frac{R^o - R^d}{k - \hat{r}^u(\tilde{Y}_k)} \end{aligned}$$

Taking the expectation of the above expression conditional on $\{\tilde{X}_1 \geq \tilde{Y}_k\}$ gives

$$\hat{R}^u = k \times E[b^*(\tilde{Y}_k) \mid \tilde{X}_1 \geq \tilde{Y}_k] > R_d$$

which was to be shown. ■

**Proof of Theorem 7**

From Milgrom and Weber (1982a, theorem 15) and the hypothesis that $(\tilde{V}, \tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n)$ are strictly affiliated, it follows that

$$\begin{aligned} D \equiv k \times E[E(\tilde{V} | \tilde{X}_1 = \tilde{Y}_k, \tilde{Y}_k) \mid \tilde{X}_1 \geq \tilde{Y}_k] - \\ k \times E[E(\tilde{V} | \tilde{X}_1, \tilde{Y}_k = \tilde{X}_1) - J(\tilde{X}_1) \mid \tilde{X}_1 \geq \tilde{Y}_k] > 0 \end{aligned}$$

where

$$J(x) \equiv \int_{\underline{x}}^{x} L(u | x)\; dt(u)$$

and where $t(u)$ and $L(u|x)$ are as defined in Equation (8).[16]

Let $M \equiv D/[kE(\tilde{X}_1 | \tilde{X}_1 \geq \tilde{Y}_k)]$. We now show that with $M$ as defined, the theorem is true. First, we recall from Lemma 2 that, with probability 1,

$$b^*(\tilde{Y}_k) > E[\tilde{V}| \tilde{X}_1 = \tilde{Y}_k, \tilde{Y}_k]$$

Therefore,

$$E[b^*(\tilde{Y}_k) | \tilde{X}_1 \geq \tilde{Y}_k] > E[E[\tilde{V}| \tilde{X}_1 = \tilde{Y}_k, \tilde{Y}_k] | \tilde{X}_1 \geq \tilde{Y}_k]$$

and

$$\begin{aligned} &k \times E[b^*(\tilde{Y}_k) | \tilde{X}_1 > \tilde{Y}_k] - \\ &k \times E[E[\tilde{V}| \tilde{X}_1, \tilde{Y}_k = \tilde{X}_1] - J(\tilde{X}_1) | \tilde{X}_1 \geq \tilde{Y}_k] > D \end{aligned} \tag{A7}$$

Next note that

$$E[\tilde{V}| \tilde{X}_1, \tilde{Y}_k = \tilde{X}_1] - J(\tilde{X}_1) = \hat{b}(\tilde{X}_1) - K(\tilde{X}_1)$$

where $\hat{b}$ is defined in Equation (7) and

$$K(x) \equiv \int_{\underset{\sim}{x}}^{x} \frac{h(u)}{f_k(u | u)} \, dL(u | x)$$

and where $h(u)$ and $L(u|x)$ are as defined in Equation (8). Next, the hypothesis that $\partial r^d(x, y_1, \ldots, y_k, p)/\partial x \leq M$ for all $x, y_1, \ldots, y_k, p$ implies that

$$k \times K(x) \leq M \times k \times x$$

Hence

$$k \times E[K(\tilde{X}_1) | \tilde{X}_1 \geq \tilde{Y}_k] \leq D \tag{A8}$$

Substituting Equation (A8) into Equation (A7) gives

$$k \times E[b^*(\tilde{Y}_k) | \tilde{X}_1 \geq \tilde{Y}_k] > k \times E[\hat{b}(\tilde{X}_1) | \tilde{X}_1 \geq \tilde{Y}_k]$$

which was to be shown. ■

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[16] Note that to prove this we also need a technical lemma that is slightly different from lemma 2 of Milgrom and Weber (1982a): Let $\rho(z)$ and $\sigma(z)$ be differentiable functions for which (1) $\rho(\underset{\sim}{x}) \geq \sigma(\underset{\sim}{x})$ and (2) $\rho(z) \leq \sigma(z)$ implies $\rho'(z) > \sigma'(z)$. Then $\rho(z) > \sigma(z)$ for all $z > \underset{\sim}{x}$. The reader should convince himself that this is indeed true.

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