American Finance Association

Price Formation and Equilibrium Liquidity in Fragmented and Centralized Markets
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Source: *The Journal of Finance*, Vol. 48, No. 1 (Mar., 1993), pp. 157-185
Published by: Blackwell Publishing for the American Finance Association
Stable URL: http://www.jstor.org/stable/2328885
Accessed: 24/01/2011 20:50

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THE JOURNAL OF FINANCE • VOL. XLVIII, NO. 1 • MARCH 1993

Price Formation and Equilibrium Liquidity in Fragmented and Centralized Markets

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ABSTRACT

This paper compares centralized and fragmented markets, such as floor and telephone markets. Risk-averse agents compete for one market order. In centralized markets, these agents are market makers or limit order traders. They are assumed to *observe* the quotes of their competitors. In fragmented markets they are dealers. They can only *assess* the positions of their competitors. We analyze differences in bidding strategies reflecting differences in market structures. The equilibrium number of dealers is shown to be increasing in the frequency of trades and the volatility of the value of the asset. The expected spread is shown to be equal in both markets, ceteris paribus. But the spread is more volatile in centralized than in fragmented markets.

THIS PAPER ANALYZES FRAGMENTED markets and compares them to centralized markets. Telephone dealer markets such as NASDAQ, SEAQ, the foreign exchange market, and the Treasury bonds market are fragmented. Examples of centralized markets are the stock and futures exchanges, such as the NYSE or the CBOT. In the latter, all the orders are addressed to the same location so that market participants can observe all the quotes and trades and take them into account in their strategies. In the former, deals are the outcome of bilateral negotiations that other market participants cannot observe. Consequently information about market conditions is more readily available in centralized markets than in fragmented markets.

This difference in market structures affects the behavior of the agents who provide liquidity to the market. Suppliers of liquidity, i.e., market makers, dealers, or limit order traders can be seen as bidders in the auction for the order flow from market order traders. The bids are the ask and bid quotes. There are two determinants of the quotes. First, they depend on the agents' private valuations of the asset. In the present paper, the agents are assumed to have the same information about the final value of the asset, but they are risk averse. Consequently, their private valuations, or their reservation

* HEC School of Management. Paper presented at the sixth world congress of the Econometric Society and the meeting of the American Finance Association, 1990. I gratefully acknowledge the helpful comments of the editor, René Stulz, the associate editor and the referee. Many thanks also to Peter Bossaerts, Patrick Bolton, Eric Briys, Michel Crouhy, Bernard Dumas, Xavier Freixas, Anne Frémault, Joel Hasbrouck, Pierre Hillion, Tom Ho, Hayne Leland, Patrice Poncet, Bob Schwartz, Raphael Shadur, Hans Stoll, and Jean-Luc Vila for stimulating discussions. Remaining errors are mine.

prices, differ according to their inventory positions.[1] Second, the bidding strategies also depend on the information sets of the agents. In particular they depend on the information each agent has about the bids or the inventories of his competitors. Because of this feature, fragmented and centralized markets differ.

Ho and Stoll (1983) analyze the case where the dealers can observe the inventories of their competitors. Using the above classification, this corresponds to centralized markets. They show that the market quote is equal to the second-best reservation quote. This is similar to English, or open, auctions.[2] This paper analyzes the opposite case, where the agents cannot observe their competitors' inventories. They only know the distribution of the inventories. In Section I, we argue that this is a reasonable assumption in fragmented markets, where the trades or the best quotes are not public information.

In Section I, the institutional characteristics of centralized and fragmented markets are discussed. In Section II, the notations, the assumptions, and the basic structure of the model are presented. In Section III, the bidding strategies in the fragmented market are analyzed. The liquidity suppliers are shown to take advantage of the lack of transparency of fragmented markets. They post ask (bid) quotes higher (lower) than their reservation quotes. This is similar to Dutch, or sealed bid, auctions. In Section IV, the market order and the equilibrium number of dealers are analyzed. The latter is such that the cost to be a dealer equals the expectation of the surplus earned by the dealers. It is shown to be increasing in the frequency of trades and the volatility of the final value of the asset. In Section V, centralized markets are analyzed and compared to fragmented markets. Although price formation differs across market structures, the expected bid-ask spread is shown to be the same. This is because the two market structures are essentially two different auctions. The present irrelevance proposition is similar to the revenue equivalence theorem obtained in the theory of auctions.[3] However, the two markets differ: the bid-ask spread is more volatile in centralized than in fragmented markets. Concluding comments are presented in Section VI. All proofs are in the Appendix.

I. Fragmented and Centralized Markets

In fragmented markets dealers stand ready to buy and sell at their bid and ask quotes. In centralized markets, specialists, market makers, or proprietary limit order traders post bid and ask quotes. As suppliers of liquidity, these three categories of agents play analogous roles, as is noted by Bronf-

[1] This is in line with the inventory paradigm of the bid-ask spread (see Stoll (1978), Amihud and Mendelson (1980), or Ho and Stoll (1983)). It differs from the adverse selection paradigm (see Glosten and Milgrom (1985), Kyle (1985)), or Admati and Pfleiderer (1988).

[2] In English auctions bidders call out ever higher bids, until only the highest bidder remains. See for instance Riley (1989).

[3] See Vickrey (1961), Harris and Raviv (1981), or Riley and Samuelson (1981).

man and Schwartz (1991). Indeed, for liquidity traders, a limit selling (buying) order is analogous to a dealer's ask (bid) quote.[4]

Centralized and fragmented markets differ in terms of information dissemination. In centralized markets, trades are the outcome of multilateral negotiations, i.e., all the agents present in the market can participate in all trades. For example, in a "floor" or a "pit," as soon as an agent quotes a price other market participants can observe it and offer a better price. Further, they can monitor the trades of their competitors and therefore their positions.[5] They can take this information into account in their own strategies. In such open outcry markets Ho and Stoll's (1983) and Ho's (1984) assumption that dealers can monitor their competitors' trades and quotes and infer their inventories is realistic. Note that this transparency can also prevail in electronic agency markets. For example, in the Paris Bourse, the five best limit-selling and -buying orders in the book are public information. Consequently, the suppliers of liquidity can undercut their competitors, by posting better orders, until their own reservation price is reached. Empirical evidence on such undercutting strategies in the Paris Bourse is provided by Biais, Hillion, and Spatt (1992). These strategies are similar to those studied by Ho and Stoll (1983).

In contrast, fragmented markets are much less transparent. Trades and quotes are often displayed on screens. But this display is generally not instantaneous. Neither is it entirely sufficient. In many OTC markets (interbank market, infrequently traded bonds or equities) firm quotes may only be obtained on the phone. Even if screen quotes are firm (which is the case in the NASDAQ, in the French government bonds (OAT) market, or in SEAQ for alpha stocks) they can be irrelevant. Deals are often the outcome of bilateral transactions negotiated on the phone, at prices within the screen quotes. The extent to which screen quotes can be improved (in terms of price or quantity) is usually uncertain. They do not reveal the intensity with which agents want to sell or buy. Therefore, in fragmented markets, the agents cannot observe the prices of their competitors. They can only assess their quotes and positions.[6] In this respect, the agents who provide liquidity to the market are

[4] These agents are different in other respects, however. In particular, their obligations are not the same. Whereas the specialist is highly regulated, proprietary traders are free to sell, to buy, or not to engage in trading. Also, market makers or members, in futures markets, must satisfy capital requirements, in contrast with proprietary traders. The model presented in this paper attempts to capture the common features of these agents. The study of their dissimilarities is left for further research.

[5] Inventory positions cannot be perfectly monitored. However, in the model presented on this paper, quotes are functions of inventory positions. Thus, agents can infer positions from price.

[6] In a recent paper, Wolinsky (1990) relies on similar insights. In particular, he remarks that, in centralized markets "trades are carried out at publicly announced prices and all traders have access to the same trading opportunities. In many important markets, however, the trading process is decentralized—prices are quoted and transactions are concluded in private meetings among agents." However, this paper differs from Wolinsky's in three respects. First, risk-averse agents are considered here, whereas Wolinsky studies risk-neutral agents, which rules out inventory considerations. Second, in this paper, there is no information asymmetry about the distribution of the final value of the asset. In contrast, Wolinsky studies the case where some agents have superior information. Third, we analyze the price formation in an auction framework, whereas Wolinsky uses a bargaining framework.

disadvantaged, compared to the general public. Market order traders can shop around market makers asking for quotes, in search of the best quotes. This is not possible for market makers. A given market maker would not reveal his best quotes, and consequently his inventory position, to a competitor asking him for a price on the telephone. This may be why special facilities are needed for interdealer trading. An example is the interdealer broker in the London SEAQ.

II. The Model

The general features of the model are discussed first. A more precise definition of the set of assumptions follows.

A. *The General Features of the Model*

Consider the market for one risky security. There are two types of agents, who supply and demand liquidity, respectively. Liquidity is demanded by outside risk-averse investors, affected by liquidity shocks. They are hereafter referred to as the *public* or the *liquidity trader*. Liquidity is provided by risk-averse agents, standing ready to trade at their bid and ask quotes. They incur a fixed cost for being present on the market. In fragmented markets, these agents are dealers and, in centralized markets, they are limit order traders or market makers. To reflect the institutional differences discussed in the previous section, it is assumed that in centralized markets the suppliers of liquidity can monitor quotes and trades, whereas in fragmented markets they do not observe the quotes or positions of their competitors. They only know the *distribution* of their competitors' inventories.

We only consider trades between the public and the suppliers of liquidity. This can be motivated in terms of risk sharing and transactions costs. Risk-sharing gains from trade arise from differences in inventories between risk-averse agents. Trades occur when these gains exceed transaction costs. Two types of costs can be noted. First, there exist taxes, order handling costs, and settlement and delivery costs. Second, strategic dealers can be reluctant to trade with their competitors, thus disclosing, at least partially, their positions. In the present model, inventory divergences between dealers are assumed to be low enough for trading costs to exceed the benefits of interdealer trading.[7] In contrast, the public is assumed to be exposed to a large liquidity shock on its risk exposure, which motivates trading with the dealers.

The entry of the liquidity suppliers in the market, and the subsequent trading process are analyzed as a game. It proceeds as follows.

1. N out of M agents decide whether to become liquidity suppliers, at a given cost.

[7] This is not explicitly modelled in the paper. For an analysis in the case of centralized markets see Ho and Stoll (1983).

2. All M agents receive inventory positions in the risky security.
3. With probability λ the liquidity shock on the risk exposure of the public occurs. In this case the public places one market order.
4. The N suppliers of liquidity compete for the order flow from the public. The buy (sell) market order is executed at the best ask (bid) price.
5. The final value of the security is realized. It is denoted P. It can be thought of as the liquidation value of the asset. At that point in time, all uncertainty about the payoff of the asset is assumed to be resolved.

The equilibrium of this game is solved for using backward induction. At Stage 4, given the size and sign of the market order, the pricing strategies of the N agents are determined. At Stage 3, the market order of the liquidity trader is determined, given her rational expectations of the pricing strategies to be followed at Stage 4. At Stage 2, the number of liquidity suppliers is determined. Their entry decision is based on their rational expectations about the market order and the pricing strategies.

B. The Specific Assumptions

The sequence of events is now described more precisely.

Stage 1: Determination of N.

First, M agents, denoted by $i = 1, \ldots, M$, can enter the market and become liquidity suppliers, i.e., dealers, market makers, or limit order traders. N agents choose to become liquidity suppliers, at a fixed nonrecoverable cost F.[8] N is determined endogenously. F accounts for the cost to stay informed about the company whose share is traded. It also reflects the cost to monitor the market, which may imply physical presence on the floor or in a trading room.[9] In the case of dealers, F also involves the cost of an administrative structure (back office) and the cost to be connected to an information network (Reuters, Telerate, etc.). Finally, in the case of specialists or futures exchanges market makers, it includes membership costs.

The M agents have identical utility functions, with Constant Absolute Risk Aversion parameter A.

$$U(x) = -e^{-Ax}, \forall x$$

Also, all agents have homogeneous expectations about the final value of the risky security.

Stage 2: Endowments.

Second, the agent i is endowed with cash, C_i, and a random inventory position, I_i. The cumulative distribution function of I_i, $F(\cdot)$ is assumed

[8] If more than one security were considered, the fixed cost could be split across securities.

[9] Liquidity suppliers using limit orders could be institutional investors, following passive trading strategies, as defined by Schwartz and Whitcomb (1988). These strategies calls for the institution to enter a mix of limit orders and to update them frequently. This indeed implies monitoring the market.

differentiable and defined on the interval $[-R, R]$ (where R is a real number).[10] For simplicity, the inventories of the different dealers are assumed to be independently and identically distributed. The interpretation of I_i, for the three categories of agents who supply liquidity to the market, is the following. (i) Dealers or specialists hold inventory positions, which result from previous trades. (ii) Proprietary traders acquired securities for investment purposes. (iii) Members in futures exchanges can hold positions in assets related to the contract traded in the exchange. For example, in the Paris futures market, the MATIF, members firms are often banks that trade in the exchange to hedge their interest rate risk exposure.

The notion of order statistics will be useful. Let $(I_i^*)_{i=1,\ldots,N}$ be the set of order statistics associated with the inventories $(I_i)_{i=1,\ldots,N}$. For example, the dealer with the longest position holds the inventory: I_N^*, the dealer endowed with the second-longest position holds the inventory I_{N-1}^*, and so on.

Stage 3: The market order.

With probability λ, the liquidity shock occurs. This is modelled as a random inventory position. More precisely, if the liquidity shock occurs, the risk-averse outside investor is endowed with a long position $+L$, with probability $1/2$, or with a short position $-L$, with probability $1/2$.[11] The liquidity trader has the same utility function as the dealers, namely exponential utility with parameter A. She also has the same expectations about the final value of the asset.

In the intuitive discussion of the motivations for trades it was argued that inventory divergences were not sufficiently large to generate trades between dealers, but were sufficiently large to generate trades between the public and the dealers. To reflect this, it is assumed that $L > R$, where R is the maximum long or short position of any dealer. Further, the liquidity shock is informationless. It is independent of the inventories of the dealers or the final value of the asset.

Given the number of suppliers of liquidity determined at Stage 1, the liquidity trader chooses the size and sign of her market order. For simplicity, it is assumed that the liquidity trader does not observe the quotes placed by the dealers. She only knows the pricing strategies to be used at Stage 4 and the distribution of inventories, $F(\cdot)$. The interpretation is that the outside investor, i.e., the liquidity trader, transmits her order to her broker without knowing what the best bid and ask prices are. Then the broker finds the best quote and transacts the prespecified quantity. This assumption simplifies the problem. It implies that, at Stage 4, the dealers worry only about the quotes of their competitors and not about the reaction of their customers to increases in the bid-ask spread. However note that the liquidity trader rationally anticipates the pricing strategies of the dealers. In particular she realizes that the bid-ask spread increases in the quantity traded, so that large buys will cost more than small buys.

[10] I_i can be positive, in which case the agent is long, or negative, in which case the agent is short. However I_i could be assumed always positive, without changing the nature of our results.

[11] The assumption that the liquidity motivation for trade is a random endowment is similar to Glosten (1989).

Stage 4: Trading.

The agent i supplies liquidity to the public by posting a buying and a selling price, denoted by A_i and B_i, respectively. Let $E(P)$ be the expected final value of the asset. A_i and B_i can be rewritten respectively as $A_i \equiv E(P)(1 + a_i)$ and $B_i \equiv E(P)(1 - b_i)$. a_i and b_i can be interpreted as selling and buying markups. For simplicity let $E(P)$ be normalized to 1.

Because of risk aversion, agents endowed with different inventory positions wish to buy or sell the risky security with different intensities. So, their bidding strategies are functions of their inventories. For simplicity, focus on a symmetric equilibrium. In this case, the buying and selling premia are: $a_i \equiv a(I_i)$ and $b_i \equiv b(I_i)$. The functions $a(\cdot)$ and $b(\cdot)$ characterize the equilibrium of the game at Stage 4. They satisfy the following condition. Given that the $N - 1$ other agents use the strategies $a(\cdot)$ and $b(\cdot)$, the agent i finds it optimal to post the buying and selling prices $E(P)(1 + a(I_i))$ and $E(P)(1 - b(I_i))$.[12] It is assumed that the agents believe that the quotes of their competitors are decreasing in their inventory positions. As shown in Proposition 1, this expectation is rational, in equilibrium.

Stage 5: The final value of the asset.

The final value of the asset, P, is realized. $P \equiv 1 + z$, where z is normally distributed with mean 0 and variance σ^2. In the case of a stock, z can be interpreted as public information eventually released about the value of the firm. z is assumed independent of the liquidity shock and the endowments of the dealers.

The extensive form of the game is represented in the tree in Figure 1.

III. Reservation Quotes and Optimal Quotes in the Fragmented Market

This section studies price formation in fragmented markets, where a given number of risk-averse dealers compete for one market order. Reservation quotes are first determined. They are such that the dealer is indifferent between trading once at these prices and not trading. They differ from optimal quotes, which maximize the surplus earned by the dealer who trades with the public. Reservation quotes are hereafter denoted by the subscript r. That is, the reservation ask quote of the dealer i is denoted by $1 + a_{r,i}$, whereas his optimal ask quote is denoted: $1 + a_i$.[13]

A. Reservation Quotes

The agent i is endowed with cash C_i and inventory I_i. If he pays the cost F but does not trade with the public, his final wealth is denoted $W_i(0)$, with:

$$W_i(0) = C_i - F + I_i(1 + z) \tag{1}$$

[12] In the centralized market, prices can be observed, so the equilibrium is simply Nash in prices. In the fragmented market, the equilibrium is Bayesian-Nash. For more formal definitions of the equilibrium concepts see Tirole (1988).

[13] Since bid and ask fees are functions of the inventory levels, a_i can also be written $a(I_i)$ and $a_{r,i}$ can be written $a_r(I_i)$.

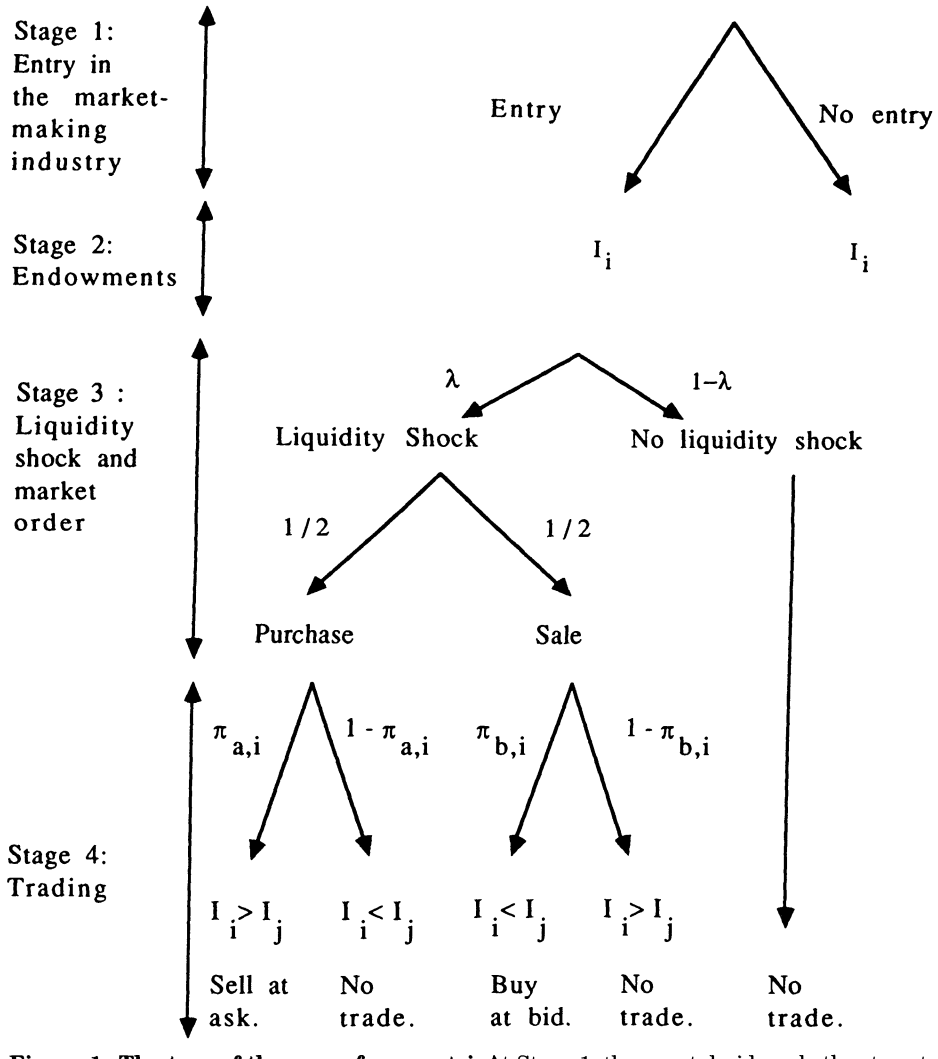

Figure 1. The tree of the game for agent *i*. At Stage 1, the agent decides whether to enter the market of not. At Stage 2, he receives his endowment in the risky asset I_i. At Stage 3, with probability λ, the public is affected by the liquidity shock, and addresses a market order to buy or sell. At Stage 4, the agent i serves the market order to buy (sell), if his inventory is larger (lower) than those of his competitors (I_{-i}). The probability that this is the case is $P(I_i > I_{-i}) \equiv \pi_{a,i}$(or $P(I_i < I_{-i}) \equiv \pi_{b,i}$).

Note that, for simplicity, the riskfree rate of interest is normalized to 0. If the agent i sells quantity Q at price $1 + a_i$, his final wealth is $W_i(a_i)$:

$$W_i(a_i) = C_i - F + I_i(1 + z) + (a_i - z)Q \tag{2}$$

Finally, if the agent i buys quantity Q at price $1 - b_i$, his final wealth is

$W_i(b_i)$:

$$W_i(b_i) = C_i - F + I_i(1 + z) + (b_i + z)Q \tag{3}$$

Reservation quotes at stage 4 of the game satisfy the following equality:

$$E(U(W_i(0))|I_i) = E_{i,4}U(W_i(a_{r,i})) = E_{i,4}U(W_i(b_{r,i})) \tag{4}$$

where $W_i(0)$, $W_i(a_{r,i})$ and $W_i(b_{r,i})$ are given in equations (1), (2) and (3), respectively. Relying on the fact that z is normal and $U(\cdot)$ is exponential negative it is easy to solve (4) for $a_{r,i}$ and $b_{r,i}$. These solutions are given in the following lemma:

LEMMA 1: *Under our set of assumptions, the reservation selling and buying prices of the dealer i (endowed with the inventory position I_i) are $1 + a_{r,i}$ and $1 - b_{r,i}$, where:*

$$a_{r,i} = a_r(I_i) = \frac{A\sigma^2}{2}(Q - 2I_i)$$

$$b_{r,i} = b_r(I_i) = \frac{A\sigma^2}{2}(Q + 2I_i)$$

$a_{r,i}$ and $b_{r,i}$ are increasing in Q. Because of risk aversion, agents have downward-sloping (reservation) demand curves.[14] Further, $a_{r,i}(b_{r,i})$ is decreasing (increasing) in inventory. The larger the inventory position of the dealer, the greater (smaller) his willingness to sell (buy). Finally, the reservation spread, $S_{r,i} \equiv a_{r,i} + b_{r,i}$, is increasing in A, the absolute risk aversion index of dealer i.

B. Optimal Quotes in the Fragmented Market

Since the problem is symmetric, only the case of the ask price is analyzed. When posting his quote, at Stage 4, the dealer i knows (i) the size of the market order, and (ii) that he will serve the order from the public if his ask price is lower than those of his competitors. However he does not know their positions, and therefore their prices. The best he can do is to compute the probability that his ask price is lower than theirs. Let this probability be denoted by $\pi_{a,i}$.[15] Before the trade, the expected utility of the dealer i is:

$$E(U(W_i(0)) + \pi_{a,i}(U(W_i(a_i)) - U(W_i(0)))|I_i) \tag{5}$$

So, the dealer solves the program:

$$\text{Max}_{a_i}\ \pi_{a,i}(E(U(W_i(a_i))|I_i) - E(U(W_i(0))|I_i)) \tag{6}$$

That is, the dealer sets his ask price to maximize the product of (i) the probability to sell and (ii) the surplus earned after a sale. Now, by the

[14] See Schwartz (1988) and Bronfman and Schwartz (1991).

[15] Similarly the probability that his bid price is higher than those of his competitors is denoted by: $\pi_{b,i}$.

definition of the reservation quotes:

$$E(U(W_i(a_i))|I_i) = E(U(W_i(0))e^{-A(a_i - a_{r,i})Q}|I_i)$$

So equation (6) is:

$$\text{Max}_{a_i}\ \pi_{a,i}(e^{-A(a_i - a_{r,i})Q} - 1)E(U(W_i(0)|I_i) \tag{7}$$

The interpretation of equation (7) is the following. In addition to his expected utility without trades $E(U(W_i(0)|I_i)$, the agent i earns, with probability $\pi_{a,i}$, a surplus from trade. The latter depends on the difference between the reservation price of the agent and the actual price at which the trade occurred. Simplifying by $E(U(W_i(0)|I_i)$, which is a negative constant, the objective of the dealer is to maximize the expected surplus from trade:

$$\text{Max}_{a_i}\ \pi_{a,i}(1 - e^{-A(a_i - a_{r,i})Q}) \tag{8}$$

It can be simplified, using Taylor expansion, and neglecting terms of the order of magnitude of $((a_i - a_{r,i})AQ)^2$

$$\text{Max}_{a_i}\ \pi_{a,i}(A(a_i - a_{r,i})Q) \tag{9}$$

The approximation does not suppress the impact of risk aversion, which determines the expected utility without trade, the reservation price, and the surplus from trade $A(a_i - a_{r,i})Q$. The approximation only linearizes the preferences of the agent over the surplus from trade. The neglected terms are likely to be small, especially if the number of dealers is large. However the approximation would be inappropriate if the amount sold was large compared to the total wealth of the dealer. This could be the case for large block trades.

From equation (9), the first order condition is:

$$\frac{\partial \pi_{a,i}}{\partial a_i}(a_i - a_{r,i}) + \pi_{a,i} = 0 \tag{10}$$

In the first-order condition, the quantity Q is not differentiated with respect to a_i. At Stage 4, the dealers take the quantity, determined at Stage 3, as given. The price has no direct impact on the quantity. The impact of the price on the quantity is only indirect. As shown in the next section, the public takes into account the pricing strategy of the dealers to determine the market order. Although the price has no direct impact on the quantity, dealer i faces the equivalent of a demand curve, like a monopolist. The monopolist faces a tradeoff: raising the price increases the profit per unit sold but it reduces the amount sold. The dealer faces a similar tradeoff: raising the ask quote increases the profit per unit sold, but it reduces the probability to sell. Therefore, the first order condition can be rewritten:

$$a(1 + (1/\epsilon)) = a_r \tag{11}$$

where: $\epsilon \equiv (d\pi_a/\pi_a)/(da/a)$. This is similar to the monopoly optimality condition, except that ϵ is not the elasticity of demand but of the probability to sell.[16]

Another interpretation of equation (10) is that the ask price is equal to the sum of the reservation price of the dealer, $a_{r,i}$, and a surplus:

$$-(\pi_{a,i}/(\partial\pi_{a,i}/\partial a_i))$$

To compute the ask price, this surplus is analyzed. In equilibrium, the dealer i expects the other dealers to use the decreasing strategy $a(\cdot)$, and finds it optimal to use the same strategy. So the probability to quote the best ask can be rewritten:

$$\pi_{a,i} = P(a_i < \mathrm{Min}(a_{-i})) = P(I_i > \mathrm{Max}(I_{-i})) \tag{12}$$

where the subscript $-i$ means everybody but i and where $P(\omega)$ denotes the probability of the event ω. Substituting equation (12) in equation (11) one can solve for a_i. Along the same lines, one can also solve for b_i. These solutions are given in the following proposition.

PROPOSITION 1: *Under our set of assumptions, the optimal bid and ask prices of the dealer i are*: $1 + a_i$ *and* $1 - b_i$, *where*

$$a_i = a_{r,i} + A\sigma^2 \frac{\int_{-R}^{I_i} F(x)^{N-1}\,dx}{F(I_i)^{N-1}} \tag{13}$$

and

$$b_i = b_{r,i} + A\sigma^2 \frac{\int_{I_i}^{R} (1 - F(x))^{N-1}\,dx}{(1 - F(I_i))^{N-1}} \tag{14}$$

The bid and ask prices are decreasing in the inventory position. Further, the surplus, $a_i - a_{r,i}$ *or* $b_i - b_{r,i}$ *is decreasing in N, and goes to* 0 *as N goes to infinity.*

When determining his optimal ask price, the agent i first takes his reservation selling price as a benchmark. At this price he is indifferent between selling and not trading. So he has an incentive to raise his ask quote above his reservation price, to capture a surplus. The larger the number of dealers, the greater the chances that raising his ask will make the dealer loose the auction, i.e., post a quote above the best of his competitors' asks. Consequently the difference between the optimal quote and the reservation price is decreasing in N. The strategic effects, generated by the relation between inventories and reservation prices, are more pronounced when the risk aversion of the dealers (A) and the variance of the final value of the asset (σ^2) are large. Therefore, the surplus earned by the dealer is increasing in

[16] Bronfman and Schwartz (1991) also show that the determination of optimal limit orders involves such as elasticity. This is because their agents also balance the magnitude of the surplus from trade and the probability to trade.

the latter. Finally, since the size of the order (Q) is known by all agents ex ante, strategic considerations are not affected by Q. Neither is the difference between the ask price and the reservation selling price.

As an illustration, the ask and bid quotes and the spread can be computed in the uniform case.

COROLLARY 1: *If $F(\cdot)$ is uniform over $[-R, R]$ then*

$$a_i = \left(-\frac{A\sigma^2}{2}(2I_i - Q)\right) + A\sigma^2\frac{(R + I_i)}{N}$$
$$= a_{r,i} + A\sigma^2\frac{(R + I_i)}{N} \tag{15}$$

and

$$b_i = \left(\frac{A\sigma^2}{2}(2I_i + Q)\right) + A\sigma^2\frac{(R - I_i)}{N}$$
$$= b_{r,i} + A\sigma^2\frac{(R - I_i)}{N} \tag{16}$$

So, the spread of dealer i is

$$S_i = a_i + b_i = S_{r,i} + 2R\frac{A\sigma^2}{N} = A\sigma^2\left(Q + \frac{2R}{N}\right)$$

IV. Equilibrium Liquidity in the Fragmented Market

A. The Optimal Market Order

At Stage 3, the liquidity shock occurs, with probability λ. In this case, the public is either endowed with a long or a short position, equal to $+L$ or $-L$, respectively. Since the problem is symmetric, only the case of a short position is analyzed. The gains from trade obtained from placing a buy order for quantity Q are:

$$[B_r(-L) - E(A(I_N^*))]Q \tag{17}$$

where the expectation is taken over I_N^*, since the dealer with the largest position serves the buy order, $B_r(-L)$ is the reservation buying price of the public, similar to the reservation prices in Lemma 1, and $A(I_N^*)$ is the best ask quote at Stage 4, given in Proposition 1.

This measure of the surplus of the public is the product of (i) the unit gain from trade and (ii) the quantity traded. It can be obtained in the same way as the objective of the dealer in equation (9). The public benefits from trading if the spread is not too large. In this case there is a $Q > 0$ such that the surplus in equation (17) is positive. The optimality of the trade and the features of the market order are analyzed in the following proposition.

PROPOSITION 2: *Under our set of assumptions, and in particular if L is larger than or equal to R, it is optimal for the public to trade with the dealers. If the liquidity trader incurred the negative liquidity shock $-L$, she addresses a market buy order for the amount*

$$Q = \frac{L + E\left(I_N^* - \dfrac{\int_{-R}^{I_N^*} F(x)^{N-1}\,dx}{F(I_N^*)^{N-1}}\right)}{2} \tag{18}$$

The interpretation of Proposition 2 is the following. The numerator of the optimal quantity in equation (18) can be split in two terms:

1. The first term is the expected inventory divergence between the public and the dealers: $L + E(I_N^*)$. The larger this expected inventory divergence, the larger the gains from trade and therefore the larger the market order.
2. The second term is proportional to the expected surplus earned by the dealers: $E((\int_{-R}^{I_N^*} F(x)^{N-1}\,dx/F(I_N^*)^{N-1}))$. The larger this surplus, the smaller the market order. Indeed when the expected surplus earned by the dealer is large, the fraction of the gains from trade retained by the public is low, which deters the placement of a large order.

$L > R$ is a sufficient condition for the first term to be larger than the second term and trading to be optimal. Further, the size of the market order is increasing in the number of market makers.[17] This result is quite intuitive, since a large number of market makers implies that the market spread is tight, other things equal.

In the case of the uniform distribution, the market order can be simply computed:

COROLLARY 2: *In the case of the uniform distribution,*

$$Q = \frac{L + \dfrac{N-3}{N+1}R}{2} \tag{19}$$

B. The Equilibrium Number of Dealers

The equilibrium is reached when no agent wishes to enter or quit the dealing industry. The equilibrium number of dealers is the greatest positive integer, N, such that the expected utility of those agents who choose to become dealers is at least as large as the expected utility of those agents who do not enter the industry. The existence and features of the equilibrium are characterized in the following proposition:

[17] This can be simply shown using the above proposition and the property in Appendix 1 which imply that: $2Q = L + E(I_{N-1}^*)$. I am thankful to the associate editor for suggesting this proof.

PROPOSITION 3: *There exists an equilibrium number of dealers if,*

$$1/2\left[R + L + \int_{-R}^{R} F(x)(F(x) - 2)\,dx\right]\left[\int_{-R}^{R} F(x)(1 - F(x))\,dx\right] > \frac{F}{\lambda A \sigma^2} \tag{20}$$

The equilibrium number of dealers is decreasing in the costs of market making (F) and increasing in the frequency of trades (λ) and the volatility of the asset (σ).

To understand why there exists an equilibrium number of dealers, under condition (20), note the following. The equilibrium number of dealers is such that the expected surplus of the dealers balances the costs of market making (F). More precisely, as shown in the proof, the equilibrium condition is:

$$\lambda Q E(a(I_N^*) - a_r(I_N^*))/N = F \tag{21}$$

The left hand side is the expected surplus of the dealers. It has three components: the average trading volume (λQ), the surplus earned per unit sold ($E(a(I_N^*) - a_r(I_N^*))$), and the ex ante probability to trade with the public ($1/N$). Before receiving their inventories, the dealers are identical. They have equal probabilities to trade with the public: $1/N$.

As shown in the proof, if condition (20) holds, then for $N = 2$ the expected surplus earned by the dealers is larger than the cost F. In this case, at least two agents have an incentive to enter the market. On the other hand, as N approaches infinity, the expected surplus earned by the dealers, given in the equilibrium condition (21), approaches 0. This is because (i) the ex ante probability to trade with the public goes to 0, (ii) the size of the market order remains finite, and (iii) the unit surplus earned by the dealer who trades with the public goes to 0. Further, the expected surplus is continuous in N. So, between 2 and infinity, there exists an equilibrium number of dealers, such that the costs and benefits of supplying liquidity are balanced.

The equilibrium number of dealers depends on the costs of market making, the volatility of the asset, and the frequency of trades in the following way: If the frequency of trades (λ) is high, the dealers have a high probability to earn the surplus from trade. λ is an exogenous measure of the trading volume and of the demand for liquidity. The equilibrium number of dealers is increasing in the latter. Obviously, the larger the cost of market making (F), the smaller the number of dealers. Therefore, if market-making costs are low, transaction costs for liquidity traders are low. F can be interpreted in terms of barriers to entry. Therefore, with regards to our results, regulating authorities, wishing to increase market efficiency, should facilitate the entry to the market-making industry. Finally, the equilibrium number of dealers is increasing in the volatility of the final value of the asset (σ^2), since, as noted above, the surplus is increasing in the sensitivity of the dealers to their risk exposure. Stoll (1978a) obtains an analogous result. Also Stoll (1978b) provides empirical evidence supporting this proposition.

The properties that N is increasing in the volatility of the value of the asset and in the frequency of trades are empirical implications of Proposition 3. Note that the form of these relations is nonlinear and can vary with the distribution of the inventories of the dealers, $F(\cdot)$. Econometric tests should take this into account.[18] Also, the volatility of the value of the asset is not easy to measure. In the present model, it is exogenous to the market microstructure. A proxy could be the standard deviation of the returns, measured over a long interval of time.

As an illustration of the proposition, the case of the uniform distribution is analyzed. For simplicity, L is set equal to R.

COROLLARY 3: *In the uniform case, if* $(\lambda A\sigma^2R^2/F) > 9$, *the equilibrium number of dealers is the integer part of the solution of*

$$\frac{\lambda A\sigma^2R^2}{F} = \frac{N(N+1)^2}{2(N-1)}$$

In this simple case, the condition under which there is an incentive for at least two dealers to enter the market is simply $(\lambda A\sigma^2R^2/F) > 9$. The equilibrium number of dealers is easy to obtain, numerically. For example, in Figure 2, the equilibrium number of dealers is plotted against the frequency of trades (λ). In this example, it is assumed that $(A\sigma^2R^2/F) = 27$, so that the minimum value of λ for which trades occur is $1/3$. As can be seen in Figure 2, N is increasing, less than linearly, with λ. This is because $(N(N+1)^2/2(N-1))$ increases more than linearly with N. The interpretation is the following. As N increases, both the surplus per unit and the probability to sell decrease, whereas the increase in the quantity is only a second-order effect. As N increases, an ever higher frequency of trades is needed to maintain equilibrium.

V. Comparison Between Centralized and Fragmented Markets

A. An Irrelevance Proposition

In centralized markets, limit order traders, specialists, and market makers post buying and selling prices. In floors or pits these prices are cried out. If a public order book exists, prices can also be entered in the book. The price formation mechanism in such markets is analogous to that described by Ho and Stoll (1983). Consider the trader with the longest inventory position and therefore the lowest reservation price. This agent can observe the bid and offers of his competitors. On floors, he can hear these. In agency markets, the current best selling and buying prices are public. In both cases the agent can undercut all his competitors. So, the market ask price is just below the

[18] Ho (1984) provides empirical support to a quadratic specification.

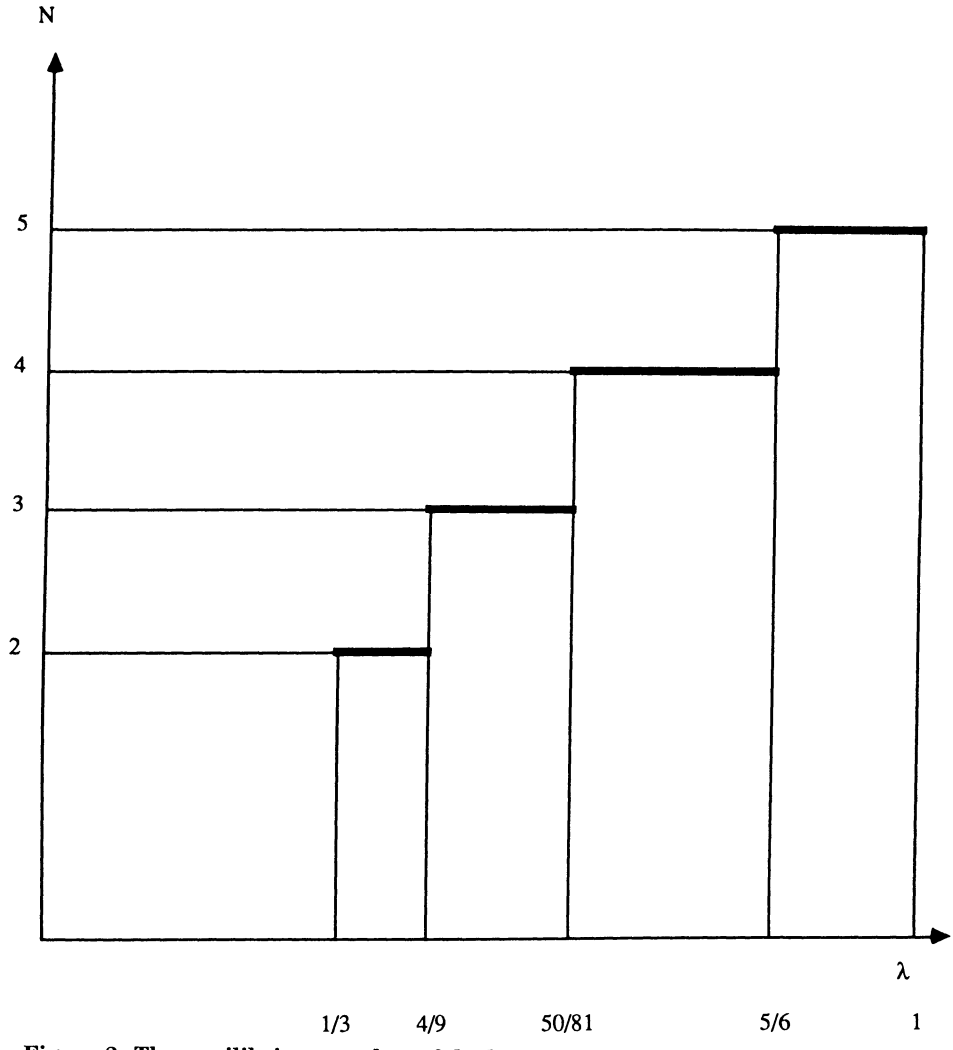

Figure 2. The equilibrium number of dealers in a simple case. Assume (i) the dealers' inventories are uniform over $[-R, R]$, (ii) the liquidity shock of the public is of the same size as the maximum dealer's position $L = R$, and (iii) the parameters that determine the number of dealers are such that

$$\frac{A\sigma^2 R^2}{F} = 27$$

where A is the Constant Absolute Risk Aversion parameter of the agents, and σ is the standard deviation of the final value of the asset. The equilibrium number of dealers (N), as a function of the frequency of trades (λ), is the step function represented by the bold line.

second-lowest reservation price. The best ask price in the centralized market, denoted a_c, is:

$$a_c = a_r(I^*_{N-1}) \tag{22}$$

Or, using Lemma 1:

$$a_c = -\frac{A\sigma^2(2I^*_{N-1} - Q)}{2} \tag{23}$$

Using equation (23) and Proposition 1, the following proposition obtains:

PROPOSITION 4: *Under our set of assumptions the expected ask (bid) price is equal in the centralized market and in the fragmented market.*

The expected ask or bid price is the same in the two market structures.[19] So too is the expected bid-ask spread. In this respect, the two markets are equally liquid. This somewhat puzzling result is analogous to the "revenue equivalence theorem" obtained in the theory of auctions. According to this theorem, the expected revenue of the seller is the same in Dutch and English auctions, if bidders are risk neutral and if their private valuations are identically and independently distributed.[20] The reason for this equivalence between auctions is the following. In the English auction the buyer with the highest private valuation bids a price just above the second-highest private valuation. In the Dutch auction, the buyer with the highest private valuation bids a price just above the second-highest private valuation. On average the two bids are equal.

This analogy between Proposition 4 and the "revenue equivalence theorem" stems from the similarity between (a) English auctions and centralized markets and (b) Dutch auctions and fragmented markets. In Dutch or sealed bid auctions, the agents do not observe the bids of their competitors. This setting is quite similar to that of fragmented markets. In contrast, English auctions are open auctions which resemble the open outcry centralized markets.

There is a difference however between our model and the setting of the "revenue equivalence theorem." In the latter, the agents are risk neutral and their different private valuations are exogenous. In contrast, in the present model, the agents are risk averse. This feature, combined with differences in inventory positions, generates differences in private valuations, i.e., reservation prices. Without the linear approximation in equation (9), risk aversion would also affect the preferences of the agents over the surplus from trade. In this case revenue equivalence would not obtain. However, under the linear approximation in equation (9), the impact of risk aversion is limited to the determination of the reservation prices. Consequently, the agents behave as risk neutral bidders with different valuations, and revenue equivalence obtains.

[19] *Expectations* are considered since the market ask and bid quotes are functions of the inventory positions of the agents, which are random variables.

[20] See the seminal paper of Vickrey (1961) or the survey paper by Riley (1989).

In the previous parts of this section the equilibrium number of liquidity suppliers was considered as given. The revenue equivalence Proposition 4 can be used to show that the equilibrium number of dealers is equal across markets. Let $N_f(N_c)$, $F_f(F_c)$ and $Q_f(Q_c)$ denote the equilibrium number of liquidity suppliers, the entry cost, and the size of the market order in the fragmented (centralized) markets, respectively.

COROLLARY 4: *Other things equal, the equilibrium number of dealers and the size of the market order are the same in the two markets.*

B. Differences Across Market Structures

Proposition 4 and Corollary 4 can be seen as a form of irrelevance proposition. Prices are not affected by the differences in market structure. However, there are two reasons why market structure could matter.

First, the ceteris paribus condition is not likely to hold. In particular, the fixed entry cost F is likely to differ across markets. To the extent that the agents must be physically present in centralized markets, the latter are more costly. This is more pronounced when the market participants are from different countries. For example, the foreign exchange market is a fragmented telephone dealer market. Still, market computerization makes it easier and less costly to centralize orders and trades. For example, in Germany, the centralized computerized market IBIS is now competing with the fragmented regional exchanges. Also, centralized markets like the NYSE or the Paris Bourse use computerized order routing, order matching, and information dissemination systems that reduce the cost of providing liquidity.

Second, although the expected bid and ask prices are the same in the two market structures, they have different distributions in fragmented and centralized markets.

PROPOSITION 5: *Other things equal, the bid-ask spread is more volatile in centralized than in fragmented markets.*[21]

The interpretation of Proposition 5 is the following. As shown above, in the centralized market, the agent with the lowest reservation price quotes his ask price just above the reservation price of his next-best competitor. In the fragmented market, he places a quote just above his expectation of this price. Proposition 5 says that this expectation is less volatile than the variable it estimates.

Mixed evidence is obtained in this subsection. Market computerization helps reduce the cost of entry in centralized markets. But the bid-ask spread is less volatile in fragmented markets. This could explain why centralized and fragmented markets coexist.

[21] I am thankful to the associate editor for the proof of this proposition.

VI. Conclusion

This paper compares centralized and fragmented markets, such as floor and telephone markets. Risk-averse agents supply liquidity to the market order from the public. In centralized markets, these agents are market makers or limit order traders, whereas in fragmented markets, they are dealers. The former can monitor the positions of their competitors but the latter can only assess these positions. The supply and demand of liquidity are analyzed as a game. The Subgame Perfect Equilibrium of this game is solved for, using backward induction.

First we study interdealer competition in fragmented markets. Each dealer exploits the fact that the other dealers do not observe his position to earn a monopolistic surplus. This surplus is decreasing in the number of dealers. Second, we analyze the optimal market order, given the rational expectations of the public about the pricing strategies of the dealers. It is increasing in the liquidity shock and in the competitiveness of the market-making system. Third, the equilibrium number of suppliers of liquidity is analyzed. It is such that the costs of market making and the expected surplus earned by the dealer are balanced. It is increasing in the frequency of trades and the volatility of the asset. This is a testable implication of the model. Finally, centralized markets are analyzed and compared to fragmented markets. The average bid-ask spread is shown to be equal in the two markets. But, the spread is more volatile in centralized than in fragmented markets, other things equal. This is another empirical implication of the analysis.

Further research could document empirically differences across centralized and fragmented markets. Theoretical research could also analyze other differences between these market structures. An issue is whether inside traders can use the lack of transparency of fragmented markets to exploit their private information. Another issue is whether the transparency of centralized markets makes it difficult for market makers to unwind their inventory positions.

Appendix 1: Properties of Order Statistics

This appendix presents certain properties of order statistics that are used in the proofs.

The c.d.f. of the jth order statistic, I_j^*, associated to the sample of N i.i.d. random variables: $\{I_i\}_{i=1,\ldots,N}$ is

$$\text{Prob}(I_j^* < x) = \sum_{k=j}^{N} C_k^N F(x)^k (1 - F(x))^{N-k}$$

where, F(·) is the c.d.f. of each of the N random variables (see Boes, Graybill, and Mood (1974)).

In the case of I_N^*, the maximum inventory position, the c.d.f. is $G_N(\cdot)$

$$G_N(I) = F(I)^N$$

So, the expected value of I_N^* is

$$E(I_N^*) = \int_{-R}^{R} I d\left(F(I)^N\right) \tag{A1}$$

Integrating by parts, this is

$$E(I_N^*) = \left[IF(I)^N\right]_{-R}^{R} - \int_{-R}^{R} F(I)^N \, dI = R - \int_{-R}^{R} F(I)^N \, dI \tag{A2}$$

In the case of the second largest inventory position, $j = N - 1$, the c.d.f. is $G_{N-1}(\cdot)$ such that

$$G_{N-1}(x) = NF(x)^{N-1} - (N-1)F(x)^N$$

So the expectation of the second largest inventory is

$$E(I_{N-1}) = \int_{-R}^{R} x \, dG_{N-1}(x)$$

integrating by parts

$$E(I_{N-1}) = [\, xG_{N-1}(x)]_{-R}^{R} - \int_{-R}^{R} G_{N-1}(x) \, dx$$

So

$$E(I_{N-1}) = R + \int_{-R}^{R} \left((N-1)F(x)^N - NF(x)^{N-1}\right) dx \tag{A3}$$

We now state the property that will be useful in the paper:

Property:

$$E(I_{N-1}^*) = E\left(I_N^* - \frac{\int_{-R}^{I_N^*} F(x)^{N-1} \, dx}{F(I_N^*)^{N-1}}\right) \tag{A4}$$

Proof

$$E\left(-\frac{\int_{-R}^{I_N^*} F(x)^{N-1} \, dx}{F(I_N^*)^{N-1}}\right) \equiv \int_{-R}^{R} \frac{\int_{-R}^{I_N^*} F(x)^{N-1} \, dx}{F(I_N^*)^{N-1}} \, dF(I_N^*)^N$$

So,

$$E\left(-\frac{\int_{-R}^{I_N^*} F(x)^{N-1} \, dx}{F(I_N^*)^{N-1}}\right) = -N\int_{-R}^{R}\left[\int_{-R}^{I_N^*} F(x)^{N-1} \, dx\right] f(I_N^*) \, dI_N^*$$

Integrating by parts, this yields,

$$-N\left[\left[\int_{-R}^{I_N^*} F(x)^{N-1} \, dx \, F(I_N^*)\right]_{-R}^{R} - \int_{-R}^{R} F(I_N^*)^{N-1} F(I_N^*) \, dI_N^*\right]$$

that is

$$-N\left[\int_{-R}^{R} F(x)^{N-1}\,dx - \int_{-R}^{R} F(x)^{N}\,dx\right] \tag{A5}$$

Using (A2) and (A5),

$$E\left(I_N^* - \frac{\int_{-R}^{I_N^*} F(x)^{N-1}\,dx}{F(I_N^*)^{N-1}}\right) = R + \int_{-R}^{R}$$

$$\left[(N-1)F(x)^{N} - NF(x)^{N-1}\right]dx = E(I_{N-1}^*) \tag{A6}$$

QED

Appendix 2: Proofs

Proof of Proposition I: The ask price is the solution of the following differential equation:

$$\frac{\partial \pi_{a,i}}{\partial a_i}(a_i - a_{r,i}) + \pi_{a,i} = 0 \tag{B1.1}$$

Now,

$$\pi_{a,i} = P(I_i > \mathrm{Max}(I_{-i})) = F(I_i)^{N-1}$$

where $-i$ denotes everybody but i. Let be $F(I_i)^{N-1}$ be denoted $G(I_i)$.

$$\frac{\partial \pi_{a,i}}{\partial a_i} = \frac{d\pi_{a,i}}{dI_i}\frac{dI_i}{da_i} = \frac{\partial G(I_i)}{\partial I_i}1/a_i'$$

where a_i' denotes the derivative of $a(I_i)$ with respect to I_i. So, the differential equation is

$$\frac{\partial G(I_i)}{\partial I_i}(a_i - a_{r,i})1/a_i' + G(I_i) = 0 \tag{B1.2}$$

or

$$\frac{\partial G(I_i)}{\partial I_i}(a_i - a_{r,i}) + G(I_i)a_i' = 0$$

or

$$\frac{\partial[G(I_i)a_i]}{\partial I_i} = g(I_i)a_{r,i} \tag{B1.3}$$

where $g(\cdot)$ denotes the derivative of $G(\cdot)$ with respect to I_i. Integrating

$$[G(x)a(x)]_{-R}^{I_i} = \int_{-R}^{I_i} g(x)a_r(x)\,dx + c \tag{B1.4}$$

where c is constant. The LHS of equation (B1.4) is zero when evaluated at $I_i = -R$. So is the integral in the RHS. So, $c = 0$. Thus, $a(\cdot)$ is such that

$$a(I_i)G(I_i) = \int_{-R}^{I_i} g(x)a_r(x)\,dx \tag{B1.5}$$

using Lemma 1 and integrating by parts the RHS of equation (B1.5), one obtains

$$a_i = a_{r,i} + A\sigma^2 \frac{\int_{-R}^{I_i} F(x)^{N-1}\,dx}{F(I_i)^{N-1}} \tag{B1.6}$$

Symmetric steps can be taken to obtain b_i.

Next, the derivative of $a(\cdot)$ with respect to I_i is analyzed. From equation (B1.6), $a_i > a_{r,i}$. From equation (B1.2),

$$a_i' = -\frac{\partial G(I_i)/\partial I_i}{G(I_i)}(a_i - a_{r,i}) \tag{B1.7}$$

The RHS of (B1.7) is negative. So $a(\cdot)$ is decreasing.

Further, the derivative of a_i with respect to N is analyzed. From equation (B1.6),

$$a_i = a_{r,i} + A\sigma^2 \int_{-R}^{I_i} \exp^{(N-1)\log(F(x)/F(I_i))}\,dx$$

So

$$\frac{\partial a_i}{\partial N} = +A\sigma^2 \int_{-R}^{I_i} \log(F(x)/F(I_i))\exp^{(N-1)\log(F(x)/F(I_i))}\,dx < 0$$

since $x < I_i$.

Finally, note that $a_i - a_{r,i}$ goes to 0 as N goes to infinity, and for all $x < I_i$, $(F(x)/F(I_i))^{N-1}$ goes to 0 as N goes to infinity. QED

Proof of Corollary 1: If, $F(\cdot)$ is uniform over $[-R, R]$, then $G(x) = (x + R/2R)^{N-1}$. So,

$$\int_{-R}^{I_i} G(x)\,dx = \frac{2R}{N}\left(\frac{I_i + R}{2R}\right)^N$$

So

$$\frac{\int_{-R}^{I_i} G(x)\,dx}{G(I_i)} = \frac{I_i + R}{N}$$

By Proposition 1,

$$a_i = a_{r,i} + A\sigma^2 \frac{I_i + R}{N}$$

QED

Proof of Proposition 2: The objective of the trader is

$$\mathrm{Max}_Q[B_r(-L) - E(A(I_N^*))]Q$$

Using Lemma 1 and Proposition 1 this is

$$\mathrm{Max}_Q\left[L - Q + E\left(I_N^* - \frac{\int_{-R}^{I_i} F(x)^{N-1}\,dx}{F(I_i)^{N-1}}\right)\right]Q$$

The first order condition is

$$Q = \frac{L + E\left(I_N^* - \dfrac{\int_{-R}^{I_i} F(x)^{N-1}\,dx}{F(I_i)^{N-1}}\right)}{2}$$

which is the value stated in the proposition. Note further that by the property in Appendix 1, this is

$$Q = \frac{L + E(I_{N-1}^*)}{2}$$

If $L > R$ this is positive, which was required for consistency, since the trade is at the ask price. Finally note that the expected surplus of the liquidity trader is

$$Q\left(E\left(I_N^* - \frac{\int_{-R}^{I_i} F(x)^{N-1}\,dx}{F(I_i)^{N-1}}\right) + L - Q\right)$$

Substituting for Q from the first order condition this is

$$\left(\frac{E\left(I_N^* - \dfrac{\int_{-R}^{I_i} F(x)^{N-1}\,dx}{F(I_i)^{N-1}}\right) + L}{2}\right)^2$$

So the surplus from the liquidity trader is positive. Hence it is optimal to trade, in spite of the bid-ask spread. QED

Proof of Corollary 2:

$$E(I_N^*) = R - \int_{-R}^{R} F(I)^N\,dI$$

Substituting $(R + I/2R)$ for $F(I)$, one obtains

$$E(I_N^*) = \frac{N-1}{N+1}R \tag{B2.1}$$

From Corollary 1, the expected surplus is $E(I_N^* + R/N)$. By equation (B2.8), this is

$$\frac{2R}{N+1} \tag{B2.2}$$

So the market order is

$$\frac{L + E(I_N^*) - \dfrac{E(I_N^*) + R}{N}}{2} = \frac{L + R\dfrac{N-3}{N+1}}{2}.$$

QED.

Proof of Proposition 3: If the agent does not decide to become a dealer, his expected utility is

$$E(U(W_i(0))\exp^{-AF}) \tag{B3.1}$$

where $W_i(0)$ is defined in equation (1). Indeed, he does not trade with the public, but he does not pay the cost F either. As can be seen from figure 1, the expected utility of any of the N dealers, after entering the market is:

$$E\left(U(W_i(0)) + \frac{\lambda}{2N}(U(W_i(a_i)) - U(W_i(0)) + U(W_i(b_i) - U(W_i(0)))\right) \tag{B3.2}$$

where $W_i(a_i)$ and $W_i(b_i)$ are defined in equations (2) and (3). Indeed, at Stage 1 inventories are unknown, so the probability to be the best seller or the best buyer is $1/N$. Further by symmetry

$$E(U(W_i(a_i)) - U(W_i(0))) = E(U(W_i(b_i)) - U(W_i(0)))$$

So equation (B3.2) can be rewritten as

$$E\left(U(W_i(0)) + \frac{\lambda}{N}(U(W_i(a_i)) - U(W_i(0)))\right) \tag{B3.3}$$

Or, by equation (7) in the text

$$E\left((U(W_i(0))) \cdot \left(1 + \frac{\lambda}{N}(-1 + \exp^{-(A(a_i - a_{r,i})Q)})\right)\right)$$

Since the agent expects to sell to the public if he holds the largest inventory position, this is

$$E\left((U(W_i(0)))\left(1 + \frac{\lambda}{N}(-1 + \exp^{-(A(a(I_N^*) - a_r(I_N^*))Q)})\right)\right) \tag{B3.4}$$

The equilibrium number of dealers is the greatest integer such that equation (B3.4) is greater than equation (B3.1), that is

$$E\left((U(W_i(0)))\left(1 + \frac{\lambda}{N}(-1 + \exp^{-(A(a(I_N^*) - a_r(I_N^*))Q)})\right)\right)$$
$$> E(U(W_i(0))\exp^{-AF}) \tag{B3.5}$$

Since z and Q are independent, equation (B3.5) can be simplified by $E(U(W_i(0)))$. So the equilibrium number of dealers is the greatest integer such that

$$E\left(1 + \frac{\lambda}{N}(-1 + \exp^{-A(a(I_N^*) - a_r(I_N^*))Q})\right) < \exp^{-(AF)} \tag{B3.6}$$

The inequality sign in equation (B3.6) has been reversed, because $E(U(W_i(0)))$ is negative. Using Taylor expansions and dropping terms of the order of $o(a(I_N^*) - a_r(I_N^*))$ and $o(F)$ as in equation (9) in the text, the inequality can be rewritten

$$\frac{\lambda}{N}E((a(I_N^*) - a_r(I_N^*))Q) > F \tag{B3.7}$$

The equilibrium number of dealers is the integer part of the solution of the following equation

$$\frac{\lambda}{N}E((a(I_N^*) - a_r(I_N^*))Q) = F \tag{B3.8}$$

This is the equilibrium condition. Using Propositions 1 and 2 it can be rewritten

$$1/N\left[E\left(I_N^* - \frac{\int_{-R}^{I_N^*} F(x)^{N-1}\,dx}{F(I_N^*)^{N-1}}\right) + L\right]/2$$

$$E\left(\frac{\int_{-R}^{I_N^*} F(x)^{N-1}\,dx}{F(I_N^*)^{N-1}}\right) = \frac{F}{\lambda A\sigma^2}$$

or, using equations (B2.6) and (B2.7)

$$\left[R + L + \int_{-R}^{R}((N-1)F(x)^N - NF(x)^{N-1})\right]/2$$

$$\left[\int_{-R}^{R} F(x)^{N-1}(1 - F(x))\,dx\right] = \frac{F}{\lambda A\sigma^2} \tag{B3.9}$$

Let the left-hand side of equation (B3.9) be denoted $\phi(N)$. The quantity purchased by the public is finite for all values of L and N. Further $1/N$ goes to 0 as N goes to infinity. Finally, from Proposition 1, $E(\int_{-R}^{I_N^*} F(x)^{N-1}\,dx)/(F(I_N^*)^{N-1}))$ goes to 0 as N goes to infinity. So $\phi(N)$ goes to 0 as N goes to infinity. Also $\phi(N)$ is continuous in N. So, if

$$\phi(2) \equiv \left[R + L + \int_{-R}^{R}(F(x)^2 - 2F(x))\right]/2$$

$$\times\left[\int_{-R}^{R} F(x)(1 - F(x))\,dx\right] > \frac{F}{\lambda A\sigma^2}$$

there exists an N, larger than or equal to 2, which solves (B3.9). At this point, two cases must be distinguished.

Case 1: If, $\phi(\cdot)$ is monotonically decreasing:

In this case, as can be seen from Figure 3, the solution of (B3.9) is unique. Further it is increasing in λ and σ and decreasing in F. The equilibrium number of dealers is the integer part of this solution.

Case 2: If $\phi(\cdot)$ is not monotonically decreasing:

In this case, although there exists at least one solution to (B3.9), it might be nonunique. But the equilibrium number of dealers is defined to be the largest integer such that the inequality (B3.7) is satisfied. So, if (B3.9) admits more than one solution, the unique equilibrium number of dealers is the integer part of the largest of these solutions. It is easy to show that in this case also it is decreasing in $(F/\lambda A\sigma^2)$. QED

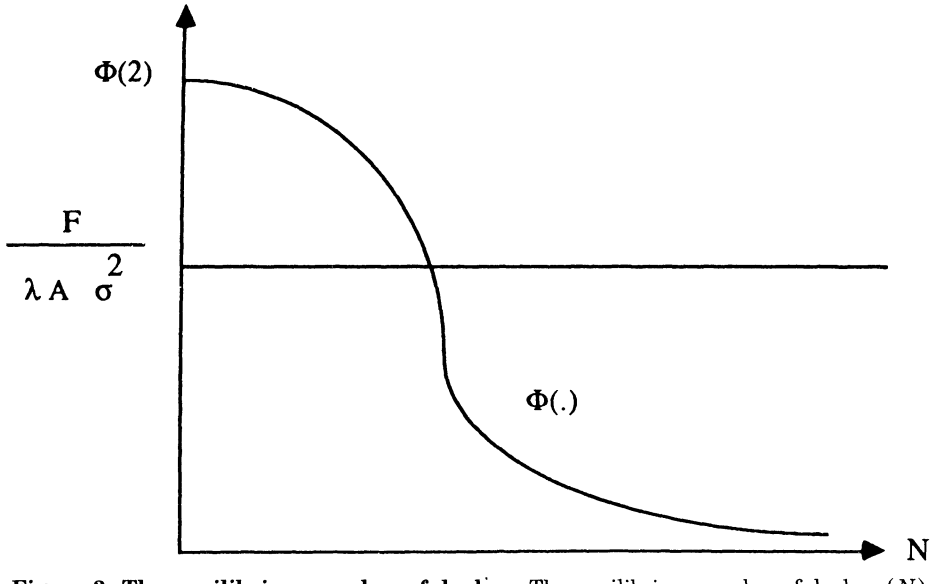

Figure 3. The equilibrium number of dealers. The equilibrium number of dealers (N) is the integer part of the solution of

$$\phi(N) = \frac{F}{\lambda A\sigma^2}$$

where A is the risk aversion of the dealers, σ is the standard deviation of the final value of the asset, F is the cost to be a dealer, and λ is the probability that there is a market order. ϕ is a continuous function that goes to 0 as N goes to infinity. If ϕ is monotonically decreasing, then there exists a unique solution to the equation. This solution is larger than 2 if

$$\phi(2) > \frac{F}{\lambda A\sigma^2}$$

Proof of Corollary 3: $L = R$, so from Corollary 2, $Q = R \cdot \frac{N-1}{N+1}$. By (B2.9), the expected surplus of the dealers is

$$\frac{2R}{N+1}$$

So the equilibrium condition (B3.8) is simplified to

$$\frac{N(N+1)^2}{2(N-1)} = \frac{\lambda A \sigma^2 R^2}{F}$$

In this case, the condition on $\phi(2)$ is simply $\frac{\lambda A \sigma^2 R^2}{F} > 9$. QED

Proof of Proposition 4: The ask price in the centralized market is

$$a_c = a_r(I^*_{N-1}) = A\sigma^2(Q/2 - I^*_{N-1})$$

The ask price in the fragmented market is from proposition 1

$$a_f = A\sigma^2\left(Q/2 - I^*_N + \frac{\int_{-R}^{I_i} F(x)^{N-1}\,dx}{F(I_i)^{N-1}}\right)$$

So the expected ask price is

$$E(a_f) = A\sigma^2\left(Q/2 - E\left(I^*_N + \frac{\int_{-R}^{I^*_N} F(x)^{N-1}\,dx}{F(I^*_N)^{N-1}}\right)\right)$$

From the property in Appendix 1 this is equal to the expectation of the ask price in the centralized market. QED

Proof of Corollary 4: To prove the corollary, we check that $N_f = N_c$, $Q_f = Q_c$ is an equilibrium. First assume $N_f = N_c$, $Q_f = Q_c$, then, by Proposition 4, the expected ask and bid prices are the same in the two markets. Second, using this result, note that, if $N_f = N_c$ then $Q_f = Q_c$. Third, write the equilibrium condition in the two markets

$$\frac{\lambda}{N_f} E_1\left(\left(a_f - a_r\left(I^*_{N_f}\right)\right)Q_f\right) = F_f$$

and

$$\frac{\lambda}{N_c} E_1\left(\left(a_c - a_r(I^*_{N_c})\right)Q_c\right) = F_c$$

Using the previous remarks and because of Proposition 4, if N^* is the solution of the first equation, it also solves the second equation. This concludes the proof. QED

Before we prove Proposition 5, we state and prove Lemma 5, which will be useful in the proof of the proposition.

LEMMA 5:

$$a(I_i) \equiv a_i = A\sigma^2[Q/2 - E(\text{Max } I_i | \text{Max } I_i < I_i)]$$

Proof: The ask price quoted by agent i in the fragmented market is:

$$a(I_i) \equiv a_i = A\sigma^2\left[Q/2 - I_i + \frac{\int_{-R}^{I_i} F(x)^{N-1}\,dx}{F(I_i)^{N-1}}\right]$$

We show below that this is equal to

$$A\sigma^2[Q/2 - E(\text{Max } I_{-i} | \text{Max } I_{-i} < I_i)]$$

First note that,

$$E(\text{Max } I_{-i} | \text{Max } I_{-i} < I_i) = \int_{-R}^{R} x d(P(\text{Max } I_{-i} < x | \text{Max } I_{-i} < I_i))$$

Now

$$P(\text{Max } I_{-i} < x | \text{Max } I_{-i} < I_i) = \frac{P(\text{Max } I_{-i} < x, \text{Max } I_{-i} < I_i)}{F(I_i)^{N-1}}$$

For $x < I_i$,

$$P(\text{Max } I_{-i} < x, \text{Max } I_{-i} < I_i) = P(\text{Max } I_{-i} < x)$$

For $x > I_i$,

$$P(\text{Max } I_{-i} < x, \text{Max } I_{-i} < I_i) = P(\text{Max } I_{-i} < I_i)$$

So,

$$E(\text{Max } I_{-i} | \text{Max } I_{-i} < I_i) = \frac{\int_{-R}^{I_i} x d(P(\text{Max } I_{-i} < x)}{F(I_i)^{N-1}} = \frac{\int_{-R}^{I_i} x d\left(F(x)^{N-1}\right)}{F(I_i)^{N-1}}$$

Integrating by parts

$$E(\text{Max } I_i | \text{Max } I_i < I_i) = I_i + \frac{\int_{-R}^{I_i} F(x)^{N-1}\,dx}{F(I_i)^{N-1}}.$$

QED

Proof of Proposition 5: From Lemma 5,

$$a_f = Q/2 - E(I_{N-1}^* | I_N^*)$$

Now,

$$a_c = Q/2 - I_{N-1}^*$$

So a_f is less volatile than a_c since

$$V(I_{N-1}^*) > V(E(I_{N-1}^* | I_N^*).$$

QED

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