Auctions of Divisible Goods: On the Rationale for the Treasury Experiment

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We compare a sealed-bid uniform-price auction (the Treasury's experimental format) with a sealed-bid discriminatory auction (the Treasury's format heretofore), assuming the good is perfectly divisible. We show that the auction theory that prompted the experiment, which assumes single-unit demands, does not adequately describe the bidding game for Treasury securities. Collusive strategies are self-enforcing in uniform-price divisible-good auctions. In these equilibria, the seller's expected revenue is lower than in equilibria of discriminatory auctions.

In September 1992, the Treasury began experimenting with a uniform-price auction for selling Treasury notes. This experimental format will be used in the monthly two-year and five-year note auctions for a period of one year. In the discriminatory auction format that has been used since the mid-1970s, the price at which any winning bid is filled is the bid price.[1] In the uniform-price auction format, each winning bid is filled at the lowest winning price (highest win-

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ning yield).[2] Early evidence indicates that the experiment has not been successful;[3] however, our purpose is not to assess the experiment but to contribute to the theory that prompted it. The experiment arose out of a general review of the Treasury's auction procedures—a review provoked by the 1991 Salomon scandal. According to the Undersecretary of the Treasury for Finance, Jerome Powell, the primary motivation for the experiment was the "very substantial academic opinion that the single price auction could result in lower financing costs."[4]

There are two sorts of academic opinion supporting the uniform-price format. There is an informal argument that collusion among bidders is less likely in a uniform-price auction. The argument is that a discriminatory auction discourages relatively uninformed bidders because of the severity of the winner's curse, so bidding becomes concentrated among a few bidders who therefore may find it feasible and profitable to collude. This argument was advanced most notably by Milton Friedman (1960, pp. 64–65):

> *(Having) different purchasers . . . pay different prices for the same security . . . establishes a strong tendency for the initial market to be limited to specialists and gives them a strong incentive to collude with respect to the bids submitted. . . . A decidedly preferable alternative is to ask bidders to submit a schedule of the amounts that they will buy at a series of prices or of coupons; to combine these bids, and set the price or coupon rate at the level at which the amount demanded equals or exceeds the amount offered.*

The second strand of academic support is founded on the theory of auctions for indivisible goods. This theory shows that a second-price auction[5] yields more revenue on average than does a first-price auc-

[1] This description conveys the essence of the auction format. The actual mechanics of note and bond auctions are as follows. A bid consists of a quantity-yield pair. The bids at lowest yields are accepted. The coupon rate is set equal to the quantity-weighted average yield of the accepted bids (to the nearest $\frac{1}{8}$ of 1 percent). The price on each winning bid is then set so that the yield equals the yield bid.

[2] This type of auction is also called a "single-price" or "Dutch" auction. The term Dutch auction has a different meaning outside financial markets.

[3] The average markup of auction yields over when-issued yields (midpoints of spreads at 1:00 P.M. on auction dates) through February 1993, has been 1.35 basis points for the two-year notes and 2.31 basis points for the five-year notes (we thank R. H. Wrightson & Associates for this data). As a point of comparison, the average markup in the 66 Treasury coupon auctions from January 1990 through September 1991 was only 0.56 basis points (Simon, 1992a). NOTE ADDED IN PROOF: The average markup for the full twelve-month experiment was 0.83 basis points for the two-year notes and 1.17 basis points for the five-year notes.

[4] This quotation comes from the Reuter transcript report of a news conference at the Treasury Department on September 3, 1992.

[5] A second-price auction is an auction for an indivisible good in which the highest bidder gets the object but pays the price bid by the second-highest bidder. In a first-price auction, the highest bidder gets the object and pays the price he bid.

tion, at least when bidders are risk neutral. The key is that the winner's curse is less severe in a second-price auction, so bidders bid more aggressively. Because of the more aggressive bidding, the second highest bid in a second-price auction is higher on average than is the highest bid in a first-price auction. Several economists have assumed this logic would extend to Treasury auctions. Referring to Treasury auctions, McAfee and McMillan (1987, p. 728) state: "theory predicts that the uniform-price auction, which is similar to the second-price auction, yields more revenue than the discriminatory auction, which corresponds to the first-price auction." Similarly, Milgrom (1989, p. 3), arguing on the basis of a model in which each bidder wants only one unit of the good being auctioned, states: "a sealed-bid Treasury bill auction in which each bidder pays a price equal to the highest rejected bid would yield more revenue to the Treasury than the current procedure in which the winning bidder pays the seemingly higher amount equal to his own bid."[6] This claim has been repeated recently by Bikhchandani and Huang (1992), Chari and Weber (1992), and Smith (1992).

The main point of this article is that the results based on single-unit demands do not generalize to auctions in which bidders desire multiple units. For a buyer of a single unit, marginal cost equals price. However, for a buyer of multiple units in a uniform-price auction, marginal cost may exceed price. This is of great importance because marginal cost is endogenous—the supply curve faced by a bidder is the residual from the demands of other bidders, so his marginal cost depends on his competitor's strategies. By submitting steep demand curves, bidders can make marginal cost much higher than price for their competitors, leading to equilibria in which value is much higher than price. In these equilibria, the seller's revenue can be much lower than the revenue obtained from a discriminatory auction, so the ranking of auction formats which holds for single-unit demands does not generalize.

Our results have some bearing on Friedman's argument. The uniform-price auction equilibria we construct could be characterized as "collusive," even though they are noncooperative equilibria. This shows that coordination and information sharing are important in uniform-price auctions as well as in discriminatory auctions. However, we do not develop a formal model of collusion.

[6] McAfee and McMillan define a uniform-price auction as an auction "in which all pay a price equal to the lowest accepted bid." This is the procedure being used by the Treasury in its experiment. Milgrom argues in favor of auctions "in which each bidder pays a price equal to the highest rejected bid." Using the lowest accepted bid or highest rejected bid can lead to different results in principle (see note 12), but in practice they are equivalent. The reason is that in Treasury auctions there is always excess demand at the market-clearing price, so the lowest accepted bid and highest rejected bid are the same. The choice of lowest accepted bid or highest rejected bid does not matter for our main result (Theorem 1).

Our results are illustrated by the following simple example. Suppose \$10 billion of notes are to be sold and there are three bidders. Suppose each bidder knows that the yield in the after-market will be 5 percent. If the notes were indivisible (i.e., if the only bids that could be submitted were for the entire \$10 billion), then the only equilibrium in either a first-price or a second-price auction would be for bidders to bid 5 percent. Consequently, the auction would be an efficient mechanism for selling the notes. However, in reality, bids do not have to be for the entire \$10 billion. Consider the following strategies: each bidder bids for \$3333 million at 6 percent and bids for \$6667 million at 20 percent. Given a uniform-price format, the entire \$10 billion will be sold at 20 percent, a very favorable yield for the bidders. Essentially, each bidder in this example has a quota of one-third of the market and is adhering to it. The point we wish to emphasize is that this "collusion" on the part of the bidders is consistent with self-interested behavior. The collusion is enforced by the steepness of the demand curves submitted by the bidders: a steep demand curve for one bidder implies a high marginal cost for his competitors, which will cause his competitors to optimally restrain their bidding. Specifically, each bidder is getting $\$3333\frac{1}{3}$ million at 20 percent. He could increase his quantity to \$3334 million by lowering the yield on his \$6667 million bid to 19.99 percent, but this would cause the yield on the entire issue to fall to 19.99 percent. Losing a basis point on $\$3333\frac{1}{3}$ million costs more in lost income than is gained from the additional $\$\frac{2}{3}$ million face value, so lowering the yield to 19.99 percent is not optimal.[7] To increase the quantity above \$3334 million will cause the yield to drop from 20 to 6 percent or below, which would certainly not be profitable, so adhering to the collusive arrangement is optimal. Note that this sort of equilibrium exists regardless of the number of bidders in the auction. Increasing the number of bidders need not increase the price received by the seller.

A discriminatory auction works much better in this example. The "collusive" equilibrium unravels in a discriminatory auction. If other bidders are bidding in the way described above, then each bidder will find it optimal to bid for the entire quantity at, say, 19.99 percent. This will increase the yield the bidder receives on \$3333 million from 6 to 19.99 percent while only dropping the yield on $\$\frac{1}{3}$ million by one basis point. Also, it will increase the quantity the bidder receives from $\$3333\frac{1}{3}$ million to \$3334 million. However, if everyone does this,

[7] This argument depends on the fact that bids are in two decimal places. However, in the formal model, we will take the price grid to be continuous and the securities to be perfectly divisible and show that strategies like this are still self-enforcing.

then each will find it optimal to bid for the entire quantity at 19.98 percent (thereby capturing the entire $10 billion), and so forth. In fact, in any pure strategy equilibrium of a discriminatory auction the yield will be 5 percent on the entire issue. Therefore, a discriminatory auction is an efficient mechanism for selling securities in this situation.

This example illustrates a key difference between uniform-price and discriminatory auctions. Relatively high inframarginal bids (steep demand curves) inhibit competition. Such bids are costless in a uniform-price auction. However, they are costly in a discriminatory auction. This cost induces bidders to submit flatter demand curves, which in turn stimulates greater price competition at the margin.

There have been very few theoretical studies of auctions for divisible goods.[8] Maskin and Riley (1989) and Branco (1993) characterize optimal mechanisms for sellers of divisible goods, but they do not compare uniform-price anad discriminatory auctions. Also, Maskin and Riley assume bidders have independently distributed valuations for the good, which is not a good model for Treasury securities. Branco allows for common values but assumes the value is additively separable in bidders' signals and signals are independently distributed, which is also not a reasonable model for Treasury securities. The most relevant article is Wilson (1979), which compares a uniform-price auction for a divisible good (a "share auction") with an auction in which the good is treated as indivisible (a "unit auction"). Wilson concludes that "a share auction is subject to manipulation by the bidders, with the result that the sale price is reduced significantly." This manipulability of uniform-price auctions is what we described as collusion. Theorem 1 is a general version of Wilson's result. The additional contribution of this paper is the comparison of uniform-price auctions with discriminatory auctions.

A few empirical articles have compared uniform-price and discriminatory auctions for divisible goods. Baker (1976) reviews the Treasury's experience with uniform-price auctions in the early 1970s. He describes his results as tentative and states that "there may be the possibility of a small cost saving to the Treasury by pricing through use of the uniform-price auctions; that is, however, probably the shakiest result of all." Simon (1992b) has recently reexamined this experience and finds that the uniform-price auctions cost the Treasury money. He finds that the markup of auction yields over when-issued yields was significantly higher when the Treasury used a uniform-

[8] In the study of multiobject auctions it has generally been assumed either that each bidder wants only one unit [e.g., Harris and Raviv (1981), Bikhchandani and Huang (1989), Milgrom (1989)] or that the objects are different or that the objects are auctioned sequentially. See Weber (1983) for a review of results in these areas and McAfee and McMillan (1987) for additional references.

price auction. This finding is consistent with our results. As mentioned, it also seems to be consistent with the early results of the current experiment. However, different results were obtained by Umlauf (1993) in a study of Mexican Treasury auctions and by Tenorio (1993) in a study of Zambian foreign exchange auctions. The Mexican Treasury switched from a discriminatory to a uniform-price format in mid-1990. Umlauf analyzes the profits of bidders in 150 discriminatory auctions and 26 uniform-price auctions. He finds that the bidders' profits dropped dramatically to near zero when the uniform-price auction was introduced. Zambia conducted a weekly auction of U.S. dollars between 1985 and 1987 and switched from a uniform-price format to a discriminatory format in 1986. Tenorio finds that average prices were higher under the uniform-price format; however, the lack of a competitive secondary market to establish benchmark prices makes these results less reliable.

The Mexican Treasury experience is puzzling. It is possible that it illustrates Friedman's point. According to Umlauf, there is open collusion among the six largest bidders in the Mexican auctions. The percentage of the aggregate issues won by these six bidders in Umlauf's sample was 73 percent under the discriminatory format and only 62 percent under the uniform-price format. It seems possible, therefore, that competition from bidders outside the cartel increased when the uniform-price format was introduced, and this led to the decline in bidder profits. Nevertheless, it seems surprising that bidder profits fell to the extent they did, given the continued importance of the six largest bidders. Another possible explanation is discussed in the Conclusion.

The model studied in this paper is as follows. A single seller wishes to sell a fixed quantity Q of a perfectly divisible good to n bidders. The bidders are assumed to possess heterogeneous information concerning the value $\tilde{v}$ of the good. We assume that there are numbers v^L and v^H such that $v^L \leq \tilde{v} \leq v^H$ with probability 1. This is without loss of generality for Treasury securities, because one can take v^L to be zero, and v^H to be the undiscounted sum of the security's payouts (tighter bounds can be obtained by considering substitute securities). For simplicity and added generality (the model applies to goods other than bonds), we take bidding to be in terms of prices rather than yields. We assume the seller sets a reserve price $p^L \geq 0$ and does not consider bids at prices below p^L. In both the uniform-price auction and discriminatory auction, bidders submit demand schedules. The price at which aggregate demand equals supply is called the *stop-out price*. If there are flats in demand curves that cause the aggregate demand to exceed supply at the stop-out price, only the marginal bids are rationed. This is the procedure used by the Treasury. In a uniform-price auction, each bidder pays the stop-out price on the quantity he is awarded. In the discriminatory auction, each bidder pays the area under his demand curve out to the quantity he receives.

In our analysis of uniform-price auctions in the general model, we assume $p^L \leq v^L$. We show that for any $p \in [p^L, v^L]$, there is a symmetric pure-strategy equilibrium in which, independent of the signals received by bidders, the price received by the seller is p. The demand curves of the bidders in these equilibria do not depend on the bidders' signals and are similar in form to the demand curves in the above example. There may be other equilibria that we have not found. If the reserve price p^L is set above v^L, then bidders may not want to buy the entire issue when they have bad signals. We analyze this case in the context of an example.

In our analysis of discriminatory auctions, we assume bidders are risk neutral. The tendency to submit relatively flat demand curves in a discriminatory auction motivates us to look for an equilibrium in which bidders submit entirely flat demand curves. We show that such an equilibrium exists.[9] The bid prices in this equilibrium are the equilibrium bids for a first-price auction in which the good is treated as indivisible. These prices are higher than v^L, so, when $p^L \leq v^L$, this equilibrium is better for the seller than are the equilibria of the uniform-price auction described in the preceding paragraph. This is the only equilibrium we have found, but we do not have a proof of uniqueness. In the example, we show that this equilibrium with an optimal reserve price yields at least as great and sometimes greater expected revenue than do the equilibria we have found for the uniform-price auction with an optimal reserve price.

The collusive equilibria we construct for uniform-price auctions rely on bidders submitting relatively high inframarginal bids. These bids are never marginal, so it is costless for bidders to submit them. If there were some randomness in demands (e.g., from noncompetitive bidders), these bids could sometimes be marginal. One would expect this to inhibit the submission of such bids and thereby reduce the degree of collusion that can be supported in equilibrium. To analyze this, we consider a model in which there are random noncompetitive bids. We assume the bidders are risk neutral and have the same information. This is not a good model of auctions, but it is tractable and allows us to examine the robustness of our general argument to randomness in demands. Our main point remains valid in this model. In any pure-strategy equilibrium of the discriminatory

[9] In Treasury auctions, there is a maximum amount for which any bidder can bid. When this constraint is present, there is an equilibrium in which each bidder submits a single bid for the maximum quantity.

auction, the seller receives the expected value of the securities, conditional on the bidders' common information. However, there are a continuum of pure-strategy equilibria of the uniform-price auction, all of which generate less, or at best the same, revenue for the seller as the discriminatory auction.

We describe the general model more fully in the following section. The main results for uniform-price auctions are presented in Section 2 and the results for discriminatory auctions in Section 3. The example is presented in Section 4. We characterize optimal mechanisms for the example. Neither the uniform price nor the discriminatory auction is optimal in general. The model with random demands is analyzed in Section 5. We conclude in Section 6 with some suggestions for future research.

1. The Model

There are $n > 1$ bidders and a single seller. The good is assumed to be perfectly divisible, and a quantity Q is to be sold. The value per unit of the good is a random variable $\tilde{v}$. Prior to the auction, each bidder observes a signal $\tilde{s}_i$ that is correlated with $\tilde{v}$. Let $\tilde{s} = (\tilde{s}_1, \ldots, \tilde{s}_n)$, and denote a generic value of $\tilde{s}$ by s. The joint distribution of $(\tilde{v}, \tilde{s})$ is assumed to be known to all bidders. Let S_i denote the support of $\tilde{s}_i$. Assume there exist numbers v^L and v^H such that $0 \le v^L \le \tilde{v} \le v^H$ with probability 1.

The seller sets a reserve price $p^L \ge 0$. A strategy of bidder i is the selection of a demand schedule for each $s_i \in S_i$. A demand schedule is a nonincreasing left-continuous function q: $[p^L, \infty) \to [0, Q]$. Denote the demand schedule of bidder i by $q_i(\cdot \mid s_i)$. The aggregate demand schedule is $q_A(\cdot \mid s) \equiv \Sigma_{i=1}^{n} q_i(\cdot \mid s_i)$.

In Section 3, we will need to consider mixed strategies. For this purpose, we assume each bidder i observes a random variable $\tilde{z}_i$ that is uniformly distributed on [0, 1] and uses this to randomize; that is, he chooses his demand schedule as a function of s_i and z_i. To ensure that the randomization is not coordinated across agents, we assume the random variables $\tilde{z}_1, \ldots, \tilde{z}_n$ are independent.

In both auction formats, all bids above the stop-out price are accepted. The stop-out price, which we denote by $p^e(s)$, is the maximum price at which demand equals or exceeds supply or it is the reserve price if there is excess supply at all prices. Formally,

$$p^e(s) = \begin{cases} \max\{p \ge p^L \mid q_A(p, s) \ge Q\} & \text{if } \{p \ge p^L \mid q_A(p, s) \ge Q\} \ne \emptyset, \\ p^L & \text{if } \{p \ge p^L \mid q_A(p, s) \ge Q\} = \emptyset. \end{cases}$$

If there is a discontinuity (flat) in the aggregate demand curve at the stop-out price, then it may be necessary to ration the demands. We assume there is pro rata rationing of marginal bids. This is defined as follows. The flat in an individual's demand curve is $\Delta q_i(p \mid s_i) = q_i(p \mid s_i) - \lim_{p' \downarrow p} q_i(p' \mid s_i)$, and the flat in the aggregate demand curve is $\Delta q_A(p \mid s) = \Sigma_{i=1}^n \Delta q_i(p \mid s_i)$. The fraction of the flat in the aggregate demand curve that cannot be filled is

$$\lambda(s) = \max\left\{\frac{q_A(p^e(s) \mid s) - Q}{\Delta q_A(p^e(s) \mid s)}, 0\right\},$$

so the quantity received by bidder i is

$$q_i^e(s) \equiv q_i(p^e(s) \mid s_i) - \lambda(s)\Delta q_i(p^e(s) \mid s_i). \tag{1}$$

In the uniform-price auction, each bidder pays the stop-out price, so his payment is $p^e(s)q_i^e(s)$. In the discriminatory auction, each bidder pays the area under his demand curve out to $q_i^e(s)$. This is given by

$$p^e(s)q_i^e(s) + \int_{p^e(s)}^{\infty} q_i(p \mid s_i)\, dp.$$

Throughout, "equilibrium" will mean Bayesian–Nash equilibrium of the auction game.

2. Uniform-Price Auctions

For our first result, we suppose the reserve price p^L is at or below the lower bound v^L of the value distribution. Theorem 1 shows that there are equilibria of a uniform-price auction that are very bad for the seller—the seller would do as well by discarding the auction format and fixing the price at v^L. If bidders are risk neutral, the theorem is actually true if v^L is replaced throughout by the minimum conditional expectation of $\tilde{v}$ given any bidder's signal (see Proposition 1 for an illustration of this). This suggests that the reserve price can be set somewhat above v^L. One could try to improve the outcome by raising the reserve price even further, but this would supplant the price-discovery role of the auction and lead to the possibility of undersubscription. This is also illustrated in Proposition 1.

Theorem 1. *Assume $p^L \leq v^L$. For each $p^* \in [p^L, v^L]$, there exists a pure-strategy symmetric equilibrium of the uniform-price auction in which $p^e(s) = p^*$ in every state s and in which the bidders' demand curves do not vary with their signals. The equilibrium demand curve is*

$$q_i(p) = \begin{cases} 0 & \text{if } p > p\dagger, \\ Q\left[\dfrac{p\dagger - p}{n(p\dagger - p) + p - p^*}\right] & \text{if } p^* < p \le p\dagger, \\ \dfrac{Q}{n-1} & \text{if } p^L \le p \le p^*, \end{cases} \tag{2}$$

where $p\dagger = (n-1)v^H/n + p^*/n$. Each bidder receives the quantity Q/n in every state s.

Proof. Suppose all bidders $j \neq i$ submit the demand curve (2) and consider the optimization problem of bidder i. The residual supply curve he faces is the total quantity Q minus the demands of other bidders. This is

$$\begin{cases} x(p) = Q & \text{if } p > p\dagger \\ x(p) = \dfrac{Q}{n}\left(\dfrac{v^H - p^*}{v^H - p}\right) & \text{if } p^* < p \le p\dagger, \\ x(p) \in \left[0, \dfrac{Q}{n}\right] & \text{if } p = p^*, \\ x(p) = 0 & \text{if } p < p^*. \end{cases}$$

As this indicates, the residual supply curve has a flat at p^*. The entire quantity is demanded by the other bidders at price p^*, so no bid below p^* can be successful, and the quantity $(n-1)Q/n$ is demanded by other bidders at prices above p^*, so at most Q/n can be obtained by bidder i at price p^*. Any quantity $q^* \in [0, Q/n]$ can be obtained by submitting a demand curve $q(\cdot)$ satisfying $\lim_{p\downarrow p^*} q(p) = q^*$. There is rationing of the bids at p^* if $\lim_{p\downarrow p^*} q(p) < Q/n$.

Obviously, if the best price for bidder i is p^*, he will want the maximum possible quantity at p^*, so his demand curve will satisfy $\lim_{p\downarrow p^*} q(p) = Q/n$. If the best price for bidder i is above p^*, bidders $j \neq i$ will not be rationed, so bidder i cannot gain by forcing aggregate demand to exceed supply at any price above p^*. Furthermore, there is no reason to submit bids at prices above $p\dagger$. Therefore, the decision problem of bidder i is to choose a price $p \in [p^*, p\dagger]$ and obtain the quantity

$$x(p) = \frac{Q}{n}\left(\frac{v^H - p^*}{v^H - p}\right).$$

Given any possible value v for the securities, and any price $p \in (p^*, p\dagger)$, we have

$$\frac{d}{dp}(v - p)x(p) = -\frac{Q}{n}\left(\frac{(v^H - p^*)(v^H - v)}{(v^H - p)^2}\right) \le 0,$$

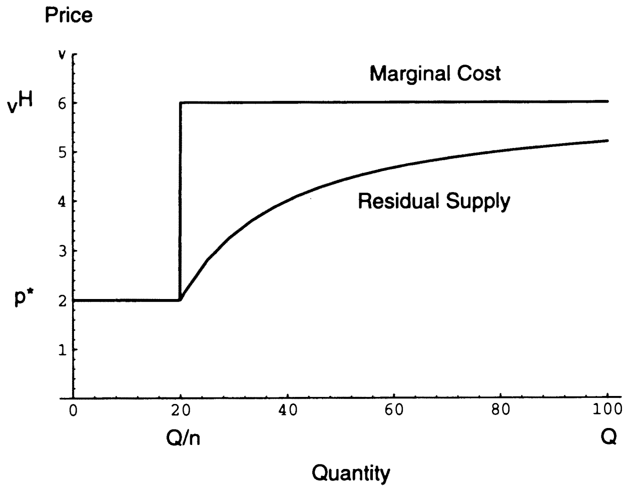

Figure 1
Residual supply and marginal cost curves facing a bidder when other bidders submit the demand curve (2), and $Q = 100$, $n = 5$, $v^H = 6$, and $p^* = 2$
Marginal cost jumps from p^* to v^H at $q = Q/n$. Therefore Q/n is the optimal quantity for any value v between p^* and v^H and therefore for any v between v^L and v^H.

because $v^H \geq v$. Hence, the best point on the supply curve is $p = p^*$, $q = Q/n$. This can be attained by submitting the demand curve (2), because $\lim_{p \downarrow p^*} q_i(p) = Q/n$. Given that this demand curve is optimal for any possible value of v, it is optimal conditional on any signal s_i. ■

Figure 1 illustrates the proof. This result is driven of course by the slope of the residual supply curve. It would also be an equilibrium for bidders to submit even steeper demand curves with flats beginning at $(p^*, Q/n)$, because this would imply a steeper supply curve and therefore a higher marginal cost.

Note that the demand for each bidder in (2) is bounded above by $Q/(n-1)$. Therefore, the demands will be feasible even when there is a maximum quantity for which any bidder can bid, as in Treasury auctions.

If other bidders do as expected—submit the demand curve (2)—then the steep portion of a given bidder's demand curve turns out to be costless; that is, these prices are never hit. It is optimal to submit this steep portion but only weakly so; for example, a flat demand curve at p^* would be just as good. It is natural to ask whether a flat demand curve at p^* would be preferred if there were some small probability that other bidders would deviate from (2). If so, we would not expect the equilibria in Theorem 1 to be realized. More generally, we would not expect the equilibria in Theorem 1 to be realized unless they are trembling-hand perfect.

The analysis of perfection is complicated by the fact that the strategy sets (the space of demand curves) are infinite dimensional. However, we can easily see that the equilibrium in Theorem 1 is trembling-hand perfect in a dense sequence of games. Take a sequence of demand curves that is dense in the space of demand curves, with (2) being the first one. Choose the sequence so that (2) is the only one that passes through the point $(Q/n, p^*)$. For $i = 1, 2, \ldots,$ construct a game by taking the strategy sets to consist of the first i demand curves in the sequence. In each game, (2) is a strict best response for each player if the other players are playing (2), so playing (2) is a trembling-hand perfect equilibrium.

This does not tell us anything about perfection in other sequences of games, such as games in which the strategy sets include the flat demand curve at p^*. The example in the introduction may shed some light on this issue. In that example, the bid at 6 percent represents the high inframarginal bid. A bidder would get the same allocation in the example by bidding for the entire issue at 20 percent, but this does not dominate bidding for \$3333 million at 6 percent and \$6667 million at 20 percent (given that the after-market yield was assumed to be only 5 percent and known to bidders). Of course, this only shows that the equilibrium is not obviously imperfect, and we would like to be able to say something stronger than this. In order to do so, we will consider a special model with nonnegligible random demands in Section 5. In this model, every portion of each bidder's demand curve can be hit with positive probability in equilibrium; yet there still exists a continuum of equilibria that yield low expected revenue for the seller.

3. Discriminatory Auctions

In a discriminatory auction, there is an obvious incentive to submit flatter demand curves than one would submit in a uniform-price auction. Assuming risk neutrality, we describe in Theorem 3 an equilibrium in which bidders submit entirely flat demand curves, bidding for the entire quantity at a single price. Bids in this equilibrium exceed v^L, so the equilibrium yields greater revenue for the seller than the equilibria of uniform-price auctions described in Theorem 1. As will be explained following the statement of the theorem, there is a similar equilibrium when bidders are prohibited from bidding for more than a given fraction of the issue.

To construct the equilibrium, we suppose that a first-price unit auction (i.e., a first-price auction in which the only bids allowed are for the entire quantity Q) has an equilibrium. In a first-price unit auction, a pure strategy for bidder i is the selection of a price $p \geq 0$ for each signal $s_i \in S_i$. Bids below p^L are allowed for convenience; such a bid is interpreted as not participating in the auction. To accommodate mixed strategies, we assume as explained in Section 1 that each bidder i observes a random variable $\tilde{z}_i$ that is uniformly distributed on [0, 1], with the $\tilde{z}_i$ being independent across bidders. A mixed strategy is a function $p(\cdot)$ of (s_i, z_i).

Theorem 2. *Assume the bidders are risk neutral. Let $(p_1, \ldots, p_n)$ denote a (possibly mixed strategy) equilibrium of the first-price unit auction. It is an equilibrium in the discriminatory auction for each bidder i to bid p_i for the quantity Q.*

Proof. By assumption, (p_i, Q) is the best bid among flat demand curves for each bidder, in response to his competitors' flat demand curves. What we need to show is that it is optimal for each bidder to submit a flat demand curve when his competitors do so. This is verified in the appendix. ■

When there are quantity constraints on bids, as in Treasury auctions, Theorem 2 must be modified. For example, suppose each bidder's demand curve must be bounded above by $Q/3$. Then there is an equilibrium in which each bidder submits a flat demand curve for $Q/3$. This equilibrium is as if there were three units being auctioned, with each successful bidder paying his bid price. The proof is the same as the proof in the Appendix, with the exception that $\bar{p}$ should be defined as the third highest price of the other bidders.

The starkest contrast between discriminatory and uniform-price auctions occurs when all bidders know the true value $\tilde{v}$.[10] Theorem 1 shows that there are pure-strategy equilibria of a uniform-price auction in which the seller receives p^*Q for any $p^* \in [p^L, \tilde{v}]$. However, in a discriminatory auction the seller receives $\tilde{v}Q$ in every pure-strategy equilibrium.

Theorem 3. *Assume each bidder knows $\tilde{v}$ (i.e., $\tilde{s}_i = \tilde{v}$ for each i). Then the seller's revenue is $\tilde{v}Q$ in any pure-strategy equilibrium of a discriminatory auction.*

Proof. Clearly the seller cannot receive more than $\tilde{v}Q$ in any equilibrium. Suppose that for some realization v of $\tilde{v}$ the seller's revenue is

[10] Equivalently, one can assume that bidders are risk neutral and have the same information. In this case, one can interpret $\tilde{v}$ as the conditional expectation of the value, given their common information.

$R < vQ$. Because revenue is greater than or equal to p^eQ, the stop-out price p^e must be less than v. Consider for any bidder i the strategy of bidding $p^e + \epsilon$ for the entire quantity Q, where $p^e < p^e + \epsilon < v$. The quantity q obtained from this strategy would be at least as great as the equilibrium quantity q_i^e, and the stop-out price would be $p^e + \epsilon$, so the profit from this strategy would be

$$(v - p^e - \epsilon)q \geq (v - p^e - \epsilon)q_i^e \rightarrow (v - p^e)q_i^e$$

as $\epsilon \downarrow 0$. Therefore the bidder's equilibrium profits must be at least $(v - p^e)q_i^e$. This implies that there is no area under the bidder's demand curve above the stop-out price p^e; that is, his demand curve is flat at p^e up to or beyond the quantity q_i^e. This must be true for all bidders. Therefore any bidder can obtain the entire quantity Q by bidding for Q at price $p^e + \epsilon$ for any $\epsilon > 0$. This would yield profits of $(v - p^e - \epsilon)Q \rightarrow (v - p^e)Q$ as $\epsilon \downarrow 0$. Therefore, each bidder's equilibrium profit must be at least $(v - p^e)Q$, which means that each bidder obtains the entire quantity, which is impossible. This contradiction establishes that the seller's revenue must be greater than or equal to vQ. ∎

Typically, there is no equilibrium in a uniform-price auction in which bidders submit flat demand curves.[11] Consider a vector $(p_1, \ldots, p_n)$ of possibly mixed strategies of flat demand curves as in Theorem 2. Suppose it is an equilibrium and that some bidder's ex ante expected profits are positive. We will show that this leads to a contradiction. Let i denote a bidder for whom there is some bidder $j \neq i$ with positive expected profits in equilibrium. Set $\bar{p} = \max\{p_j \mid j \neq i\}$. The sum of the ex ante expected profits of the bidders $j \neq i$ is

$$E[\tilde{v} - \bar{p} \mid \bar{p} \geq p_i] \cdot \text{prob}(\bar{p} \geq p_i) > 0.$$

Hence, for some values s_i of bidder i's signal,

$$E[\tilde{v} - \bar{p} \mid \bar{p} \geq p_i, \tilde{s}_i = s_i] > 0.$$

For these signals, bidder i could increase his profit by bidding very high, say $p \geq v^H$, for $Q - \epsilon$ and bidding p_i for the remaining quantity ϵ. Using this alternative strategy will not change the distribution of the winning price, but it will ensure that bidder i receives a greater quantity at the winning price at times when a greater quantity is desirable. Thus, the flat demand curves $(p_1, \ldots, p_n)$ cannot form an equilibrium.[12]

[11] An exception is when all bidders know the true value. Then, as remarked above, one of the equilibria described in Theorem 1 has all bidders bidding the true value.

[12] John Nachbar pointed out to us that this reasoning depends on the fact that the selling price in the uniform-price auction, as we (and the Treasury) define it, is the lowest winning bid rather than the highest rejected bid.

With risk aversion, it is probably not an equilibrium for bidders to submit flat demand curves in a discriminatory auction. When bidders are risk averse, their true demands are downward sloping, and one would expect this to be reflected in their bids. However, the point that discriminatory auctions induce some degree of price competition seems robust, even if the price competition is not always identical to that in a first-price unit auction.

4. An Example

Suppose there are two risk-neutral bidders who receive independent signals. Each bidder's signal takes on one of two values, which we label L and H. These values are equally likely. Let $v^L < v^H$, set $v^M = (v^L + v^H)/2$ and assume

$$\tilde{v} = \begin{cases} v^L & \text{if } \tilde{s}_1 = L \text{ and } \tilde{s}_2 = L, \\ v^H & \text{if } \tilde{s}_1 = H \text{ and } \tilde{s}_2 = H, \\ v^M & \text{otherwise.} \end{cases} \tag{3}$$

The interpretation for Treasury auctions is that each bidder receives a signal about the level of demand in the after-market, through the orders received from customers. These orders are independent and together determine the level of demand and hence the price in the after-market.

We first consider the uniform-price and discriminatory auctions with various reserve prices, constructing equilibria similar to those described in Theorems 1 and 3. Then the seller's expected revenue is computed under an optimal mechanism. Finally, we demonstrate that from a risk-neutral seller's point of view, considering only equilibria like those described in Theorems 1 and 3 and using optimal reserve prices in the auctions, the mechanisms are ranked as

$$\text{Optimal} \succeq \text{Discriminatory} \succeq \text{Uniform Price}.$$

Whether these rankings are weak or strict depends on the size of v^H relative to v^L. The precise results are

$$\begin{array}{ll} \text{Optimal} \succ \text{Discriminatory} \sim \text{Uniform} & \text{if } v^H/v^L \le 2, \\ \text{Optimal} \succ \text{Discriminatory} \succ \text{Uniform} & \text{if } 2 < v^H/v^L < 3, \\ \text{Optimal} \sim \text{Discriminatory} \succ \text{Uniform} & \text{if } v^H/v^L \ge 3. \end{array}$$

The most reasonable case is $v^H/v^L \le 2$, because the spread of possible values for Treasury securities is actually quite small. In this case the discriminatory and uniform-price auctions, each with an optimal reserve price, yield the same expected revenue for the seller, namely $(3v^L/4 + v^H/4)Q$. This revenue could be obtained simply by posting a price of $3v^L/4 + v^H/4$. This is the conditional expected value of $\tilde{v}$ for a bidder with a low signal, so all bidders would be willing to demand Q at this price. This is also the optimal reserve price in the uniform-price auction. The advantage of a discriminatory auction in this example is that it does not require the seller to compute this value. The seller can post any reserve price $p^L \leq v^L$ in the discriminatory auction, and the expected revenue will be $(3v^L/4 + v^H/4)Q$.

4.1 Uniform-price auctions

If the seller sets a reserve price $p^L \leq v^L$, then Theorem 1 shows that for each p^* between p^L and v^L there is an equilibrium in the uniform-price auction in which the entire quantity is always sold at price p^*. The equilibrium demand schedule for each type of bidder is

$$q(p) = \begin{cases} 0 & \text{if } p > p\dagger, \\ Q\left[\dfrac{p\dagger - p}{2(p\dagger - p) + p - p^*}\right] & \text{if } p^* < p \leq p\dagger, \\ Q & \text{if } p^L \leq p \leq p^*, \end{cases} \tag{4}$$

where $p\dagger = (p^* + v^H)/2$. This is actually true for all reserve prices less than or equal to $(3v^L + v^H)/4$. If the reserve price is set above $(3v^L + v^H)/4$, then there is a possibility of undersubscription. Part 1 of the following extends Theorem 1 to reserve prices up to $(3v^L + v^H)/4$, and parts 2–4 deal with higher reserve prices and undersubscription. The last statement of the proposition is a general characterization of the pure-strategy equilibria.

Proposition 1.

1. *If $0 \leq p^L \leq (3v^L + v^H)/4$, then for each $p^* \in [p^L, (3v^L + v^H)/4]$, it is an equilibrium for each type of bidder to submit the demand schedule (4). The equilibrium price is p^*, and the equilibrium demand is Q, independently of the bidders' signals. Therefore, the expected revenue is p^*Q.*
2. *If $(3v^L + v^H)/4 \leq p^L \leq v^M$, then, for $p^* = p^L$, it is an equilibrium for a low-type bidder to demand zero and a high-type bidder to submit the demand schedule (4). The equilibrium price is p^L for any signals of the buyers, and the equilibrium demand is Q if at least one of the bidders is a high type and 0 if both bidders are low types. Therefore, the expected revenue is $3p^LQ/4$.*
3. *If $v^M \leq p^L \leq (v^L + 3v^H)/4$, then for $p^* = p^L$, it is an equilibrium for a low-type bidder to demand zero and a high-type bidder to submit the demand schedule*

$$q(p) = \begin{cases} 0 & \text{if } p > p\dagger, \\ Q\left[\dfrac{p\dagger - p}{2(p\dagger - p) + p - p^*}\right] & \text{if } p^L \leq p \leq p\dagger. \end{cases} \tag{5}$$

The equilibrium price is p^L, independently of the bidders' signals. The equilibrium demand is Q if both bidders are high types, $Q/2$ if exactly one bidder is a high type, and 0 if both bidders are low types. Therefore, the expected revenue is $p^L Q/2$.

4. If $(v^L + 3v^H)/4 \leq p^L$, then it is an equilibrium for each type of bidder to demand zero.

Given any reserve price $p^L > v^L$ and any symmetric pure strategy equilibrium of a uniform-price auction, if the equilibrium demand is Q in every state, then the equilibrium price is the same in every state and no greater than $(3v^L + v^H)/4$.

Proof. See the Appendix. ■

Given equilibria as described in the proposition, the seller should set the reserve price above v^L. If $v^H/v^L \leq 3$, the optimal reserve price is $(3v^L + v^H)/4$ and the expected revenue is $(3v^L/4 + v^H/4)Q$. If $v^H/v^L > 3$, the optimal reserve price is v^M and the expected revenue is $(3v^L/8 + 3v^H/8)Q$. The motivation for using the higher reserve price v^M in the latter case is that v^L is so low it is optimal to charge high types more and not sell to low types.

A property of the equilibria in cases 1–4 is that the price received by the seller does not increase when the bidders have good signals. The last statement of the proposition shows that this is a general property of all pure-strategy equilibria, when $p^L > v^L$ and the entire quantity is always sold. In such cases, the auction does not succeed in stimulating competition among the bidders in order to arrive at a good price.

4.2 Discriminatory auctions

In a discriminatory auction a high-type bidder bids more than a low-type bidder, so the auction does succeed in eliciting information from bidders. The revenue of the seller is higher when bidders have good signals. This is true, at least, in the equilibria of Theorem 3. The following describes these equilibria for reserve prices $p^L \leq v^M$. It shows that the seller obtains nonnegligible revenue, even if the reserve price is zero. If the reserve price is set above v^M, then there are pure-strategy equilibria in which the low type does not bid and the high type bids for $Q/2$ at a single price. These are never optimal for the seller.

Proposition 2. *Suppose $p^L \le v^M$. There exists a mixed-strategy equilibrium in which the high-type bids for the entire quantity at a single price. Let $l = \max\{p^L, v^L\}$. The price bid by the high type is chosen randomly from the interval $(l, l + v^H)/2]$ according to the distribution function $F(p) = (p - l)/v^H - p)$.*

1. If $p^L \le v^L$, it is an equilibrium for the low type to randomize in an arbitrary way over quantities less than or equal to Q at price v^L. If the low-type bids for Q at this price, then the expected revenue to the seller is $(3v^L/4 + v^H/4)Q$.

2. If $p^L > v^L$, the low type does not bid. The expected revenue to the seller is $(p^L/2 + v^H/4)Q$.

There are no other equilibria in which bidders bid for the entire quantity Q at a single price.

Proof. See the Appendix. ∎

If $v^H/v^L \le 2$, any reserve price $p^L \in [0, v^L]$ is optimal and the expected revenue is $(3v^L/4 + v^H/4)Q$. If $v^H/v^L > 2$, the optimal reserve price is v^M and the expected revenue is $(v^L/4 + v^H/2)Q$.

It is interesting to compare the discriminatory auction with a second-price unit auction. It is straightforward to show that equilibrium bids in the second-price unit auction are v^L for low types and v^H for high types. The expected revenue of the seller is $(3v^L/4 + v^H/4)Q$, the same as in case 1.

4.3 Optimal mechanisms

To find an optimal mechanism it suffices, by the Revelation Principle (Myerson, 1979), to consider only direct mechanisms in which it is incentive compatible for each bidder to truthfully reveal his signal. Let x^{ab} denote the amount of the good received, and let t^{ab} denote the amount of money paid (not the price per unit) by a bidder who announces he is type a when the other bidder announces he is type b. We assume the seller's objective is to maximize the expected transfers[13] $\frac{1}{2}(t^{LL} + t^{LH} + t^{HL} + t^{HH})$ subject to selling no more than Q of the good; that is,

$$x^{LL} \le Q/2, \tag{6}$$

$$x^{LH} + x^{HL} \le Q, \tag{7}$$

$$x^{HH} \le Q/2, \tag{8}$$

and subject to the incentive compatibility constraints

[13] The assumption that the seller values only the expected transfers implies that the seller's valuation of any unsold quantity is zero. It is straightforward to modify the analysis to handle the more realistic case in which the seller's value is a nonzero constant.

$$x^{LH}v^{M} - t^{LH} + x^{LL}v^{L} - t^{LL} \geq x^{HH}v^{M} - t^{HH} + x^{HL}v^{L} - t^{HL}, \tag{9}$$

$$x^{HH}v^{H} - t^{HH} + x^{HL}v^{M} - t^{HL} \geq x^{LH}v^{H} - t^{LH} + x^{LL}v^{M} - t^{LL}, \tag{10}$$

and the participation contraints

$$x^{LH}v^{M} - t^{LH} + x^{LL}v^{L} - t^{LL} \geq 0, \tag{11}$$

$$x^{HH}v^{H} - t^{HH} + x^{HL}v^{M} - t^{HL} \geq 0. \tag{12}$$

The left-hand sides of (9) and (10)—which are repeated in (11) and (12)—equal twice the expected profit of a low type and high type, respectively, when the bidder announces his true type. The right-hand sides are twice the expected profits when bidders announce the wrong types. The participation constraints guarantee at least zero expected profit to each type. The transfers are allowed to be negative (i.e., the seller pays the bidders), but there are optima in which all transfers are nonnegative.

Proposition 3. *In an optimal mechanism, $x^{LH} = 0$, $x^{HL} = Q$, and $x^{HH} = Q/2$. Furthermore,*

$$t^{LL} + t^{LH} = v^{L}x^{LL}, \tag{13}$$

and

$$t^{HL} + t^{HH} = \left(\frac{v^{L}}{2} + v^{H}\right)Q + (v^{L} - v^{H})\frac{x^{LL}}{2}, \tag{14}$$

so the expected revenue is

$$\left(\frac{v^{L}}{4} + \frac{v^{H}}{2}\right)Q + \left(\frac{3v^{L}}{4} - \frac{v^{H}}{4}\right)x^{LL}.$$

If $3v^{L} \leq v^{H}$, then $x^{LL} = 0$ yields an optimum. If $3v^{L} \geq v^{H}$, then $x^{LL} = Q/2$ yields an optimum.

Proof. See the Appendix. ■

Notice that the high-type bidder gets the good when one type is high and the other is low. This is the same outcome that occurs in the discriminatory auction but is very different from the outcome in the uniform-price auction.

When $v^{H}/v^{L} \leq 3$, the maximum expected revenue is $(5v^{L}/8 + 3v^{H}/8)Q$. In this case it is optimal to always sell all of the good. An example of an optimal mechanism in this case is a first-price auction in which bids are for the entire quantity and in which bidders are only allowed to bid one of the prices v^{L} and v^{M}. A high-type bidder will be indifferent between the two prices and a low-type bidder will strictly prefer bidding v^L. If these bids are made, then the allocation and transfers will satisfy the conditions in the proposition.

When $v^H/v^L > 3$, the maximum expected revenue is $(v^L/4 + v^H/2)Q$. In this case it is optimal to never sell the good to a low type. This allows the seller to extract more revenue from the high type (it relaxes the incentive compatibility constraint for the high type). In fact, the seller extracts all the surplus from the high type in this case. A discriminatory auction with reserve price v^M is optimal in this case.

When $v^H/v^L < 3$, neither the discriminatory nor the uniform-price auction is optimal for the seller. The shortfall in the discriminatory auction stems from two sources: sometimes failing to sell the good when it is optimal to do so, and giving the high-type bidder excess expected profits. If $2 < v^H/v^L < 3$, then an optimal mechanism always sells all of the good, but the discriminatory auction (with optimal reserve price $p^L = v^M$) does not sell the good if both bidders have low signals. If $v^H/v^L < 2$, then both mechanisms sell the good to low-type bidders, but a high-type bidder earns excess expected profits of $(v^H - v^L)Q/8$ under a discriminatory auction. This is the shortfall in expected revenue to the seller in a discriminatory auction relative to an optimal mechanism.

5. Noise

It is unreasonable to suppose that all of the bidders in a Treasury auction will be able to coordinate on an equilibrium of the type described in Theorem 1. It is more likely that coordination will be limited to a few dominant bidders. From the viewpoint of these bidders, the bids of the other bidders will be random. In particular, competitive bidders may view noncompetitive bids as random. We show in this section that this type of noise reduces the set of equilibria of uniform-price auctions, but there are still equilibria of such auctions that are inferior to equilibria of discriminatory auctions.

We demonstrate this only for a special case, because we have not been able to solve a general model. We assume in this section that the true value v of the good is known to all of the bidders. As was true for Theorem 2, this can be interpreted in a somewhat more interesting way. If the bidders are risk neutral and have the same information, it is not necessary that they know the true value. In this case, we can interpret v as the conditional expectation of the value given their common information. Also, we will assume that the random bids are noncompetitive bids, so the aggregate supply is a random quantity and is not price sensitive. However, this random supply can have full support in $[0, Q]$ (where Q continues to denote the quantity being sold by the seller), so any portion of any bidder's demand curve can be hit with positive probability in equilibrium.

Theorem 2 remains valid when the supply is random, so the seller's revenue will be vQ in any pure-strategy equilibrium of a discriminatory auction. We will show that Theorem 1 also generalizes in the sense that there is a continuum of pure-strategy equilibria of a uniform-price auction, all but one of which generate less revenue for the seller than does the discriminatory auction.

Let $\tilde{\xi}$ denote the random demand. Continue to denote the number of strategic bidders by n and, to simplify the notation, let β denote the constant $1/(n-1)$.

Theorem 4. *Assume $p^L \leq v$. For each α such that*

$$Q(v - p^L)^{-\beta}/n \leq \alpha \leq \infty,$$

it is an equilibrium for each bidder to submit the demand curve

$$q(p) = \begin{cases} 0 & \text{if } p > v, \\ \alpha(v-p)^\beta & \text{if } p^L \leq p \leq v \text{ and } \alpha(v-p)^\beta \leq Q, \\ Q & \text{if } p^L \leq p \leq v \text{ and } \alpha(v-p)^\beta > Q. \end{cases} \quad (15)$$

The equilibrium stop-out price is

$$p(\xi) = v - \left(\frac{Q - \xi}{\alpha n}\right)^{n-1}. \quad (16)$$

The meaning of (15) for $\alpha = \infty$ is that the demand curve is flat at price $p = v$ (i.e., the quantity Q is demanded at each price less than or equal to v). This is the best equilibrium for the seller and generates the same revenue as the discriminatory auction; however, there is no mechanism by which the seller can ensure this equilibrium will occur. Since the bidders are the players in the auction game, it is more reasonable to assume their preferred equilibrium will be played. Their preferred equilibrium is at the lower bound for α. In the case $n = 2$ the expected stop-out price for this value of α takes the simple form

$$\frac{E[\tilde{\xi}]}{Q}v + \left(1 - \frac{E[\tilde{\xi}]}{Q}\right)p^L.$$

Recall that the stop-out price in the bidders' preferred equilibrium in the fixed-supply model is p^L. Therefore, the randomness of supply increases the seller's expected revenue. However, this expected revenue can still be substantially less than is provided by a discriminatory auction.[14]

[14] One difference between this model and the fixed-supply model is that the expected revenue in the bidders' optimal equilibrium in this model does converge to vQ as $n \to \infty$.

Proof. Given the strategies (15), each bidder faces the random supply curve $S(p, \tilde{\xi}) = Q - \tilde{\xi}$ if $p > v$ and

$$S(p, \tilde{\xi}) = \max\{Q - \tilde{\xi} - (n - 1)\alpha(v - p)^{\beta}, 0\}$$

if $p^L \leq p \leq v$. For each realization ξ, the optimal price on this supply curve is given by (16) with corresponding quantity $q = (Q - \xi)/n$. This price-quantity pair can be realized by submitting the demand curve (15). Therefore, this demand curve is optimal for each realization of $\tilde{\xi}$ and hence is optimal in expected value terms. ■

The demand curve (15) is optimal for each bidder because it passes through the optimal point on the residual supply curve faced by the bidder for each realization of $\tilde{\xi}$. The equilibria in the models of Kyle (1989) and Klemperer and Meyer (1989) also have this property. As in Klemperer and Meyer (1989, Section 3) one can show, under some mild regularity conditions, that the equilibria in Theorem 4 are the unique symmetric pure-strategy equilibria having the property that bidders' demand curves "pass through ex-post optimal points."

The fixed-supply and random-supply models with known v are analogous to the certainty and uncertainty models of Klemperer and Meyer. Klemperer and Meyer show that any price can be supported as an oligopoly equilibrium if demand is certain and the strategies of firms are demand curves. This is analogous to our Theorem 1. Uncertainty, in the form of a random supply in our model, reduces the set of equilibria. An important difference between our results and Klemperer and Meyer's is that there remains a continuum of equilibria in our model when the supply is random. The reason for this is that supply is bounded above by Q—the nonstrategic traders can only buy and cannot sell at the auction. If supply were unbounded above, one could show, following the reasoning of Klemperer and Meyer, that there is a unique equilibrium (in which demand curves pass through ex post optimal points). This unique equilibrium would be the $\alpha = \infty$ case, the best equilibrium for the seller. This suggests that the seller could benefit by randomizing the quantity offered.

6. Conclusion

A principal point of this article is that auctions for divisible goods are very different from auctions for indivisible goods, or, more generally, auctions in which each bidder wants only one unit of the good. Bidders buying multiple units are concerned with marginal cost rather than price. This can have great importance in auctions, because marginal cost is endogenous, being determined by the demand schedules submitted by bidders. By submitting very steep demand curves, bidders in uniform-price auctions can make marginal cost very high for other bidders and thereby inhibit competition from them. Via this device, bidders can realize a collusive outcome in a noncooperative equilibrium. This problem does not necessarily diminish as the number of bidders is increased. Furthermore, it appears to be robust to noise in the demands of bidders.

This point has practical implications for the assessment of the experiment's success. One measure the Treasury plans to consider in evaluating the experiment is the spread of winning bids. Apparently, the assumption is that a greater spread will indicate that bidders are bidding their "true values." However, as Theorem 1 and our example in the introduction indicate, a large spread can also be evidence of manipulation. The coverage ratio (the total of the quantity bids divided by the size of the issue) is another variable the Treasury will examine. This is also of doubtful value. In our example in the introduction, the coverage ratio is the maximum possible (three), yet the bidding is really not competitive.

The second main point of the paper is that uniform-price auctions may be worse than discriminatory auctions. Our results here are qualified by the fact that we have only isolated certain classes of equilibria for the two auction formats. There may be other equilibria for which the ranking of the auctions is reversed.

We have only analyzed Nash equilibria of one-shot auctions. If the bidders compete repeatedly, as they do in Treasury auctions, then the theory of infinitely repeated games tells us that many different outcomes are possible. For example, collusion can be supported by appropriate threats of retaliation against defectors. There is no theory indicating whether this is more likely to occur in a discriminatory auction or a uniform-price auction. However, it is obviously easier to enforce collusion when bidders do not have an incentive to defect in the one-shot game. This suggests that collusion based on the equilibria in Theorem 1 is likely in repeated uniform-price auctions.

A policy that might be effective for the seller in a uniform-price auction is to choose the quantity after the bids are submitted. This would allow the seller to "pick off" any high "inframarginal" bids that are submitted, so it might eliminate the collusive equilibria we have found. The Mexican Treasury has this option, which could explain Umlauf's results. We are currently studying this issue.

The presence of a pool of potential bidders is an important aspect of the auction that has not been captured in this article. The low prices in the equilibria of Theorem 1 should attract additional bidders, which may make the low prices difficult to sustain. This is related to Friedman's point that increasing the number of bidders will lead to higher prices for the Treasury. Offsetting Friedman's point is the observation that additional bidders will not be drawn to the auction unless prices are low, because the when-issued market offers a reasonable alternative to the auction.[15] Practitioners seem to be skeptical of the argument that the change in format will increase the number of bidders. Nevertheless, determining what outcomes are stable when the number of bidders is endogenous is an important subject for future research.

Another issue in need of study is the subject of open auctions. The Treasury proposed such a format but decided to shelve the idea temporarily, at least until the experiment with sealed-bid uniform-price auctions is complete. When collusion is not considered and only a single unit is being auctioned, ascending-price open auctions are better for the seller than sealed-bid first-price auctions, sealed-bid second-price auctions, or descending-price open auctions (Milgrom and Weber, 1982). However, open auctions may be more susceptible to collusion (Milgrom, 1986). Ascending price divisible-good auctions are discussed by Menezes (1993).

We have assumed that the value of the securities does not depend on the allocation in the auction. However, the postauction market in Treasury securities is not perfectly competitive, and prices in this market may depend on the auction allocation and prior when-issued trading [as the short-squeeze incidents illustrate; see Jordan and Jordan (1993)]. In general, it is important to analyze the interaction of the when-issued market with the auction.

Appendix

Proof of Theorem 2. Suppose each bidder $j \neq i$ bids p_j for the entire quantity Q. Consider an arbitrary downward-sloping demand curve. We need to show that there is some constant price p such that a bid for the entire quantity at price p is as good for bidder i as the downward-sloping curve. This will imply that an optimum among flat demand curves is optimal among all demand curves, so it will show that (p_i, Q) is an optimal response in the discriminatory auction.

Define $\bar{p}(z_{-i}, s_{-i}) = \max_{j \neq i} p_j(z_j, s_j)$. Given a demand curve $q(\cdot \mid s_i)$, the equilibrium price is $p^e(z_{-i}, s) = \max\{p^L, \bar{p}(z_{-i}, s_{-i}), \hat{p}_i(s_i)\}$, where $\hat{p}_i(s_i) = \max\{p \mid q(p \mid s_i) \geq Q\}$.

The demand curve $q(\cdot \mid s_i)$ can be approximated by step functions. Fix an arbitrary $s_i \in S_i$, denote $q(\cdot \mid s_i)$ by $q(\cdot)$, denote the kth approximating step function by q^k, and let $\hat{p}_i^k$ be defined from q^k in the same way that $\hat{p}_i$ is defined from q. We can choose q^k so that $\hat{p}_i^k = \hat{p}_i$. This

[15] This point seems to be related to Rock's (1986) explanation of IPO underpricing as a premium necessary to compensate uninfored bidders for the winner's curse.

implies that the equilibrium price is the same in all states under q^k as under q. We can also ensure that for all $1/k \leq p \leq k$, $| q^k(p) - q(p) | \leq 1/k$. This guarantees that the expected profit from q^k converges to the expected profit from q as $k \to \infty$. So it suffices to show that a flat demand curve is optimal among step functions.

We will consider a function with a single step. The argument for an arbitrary number of steps is analogous. Consider prices $p_a > p_b$ and an arbitrary quantity q. Define

$$q(p) = \begin{cases} 0 & \text{if } p > p_a, \\ q & \text{if } p_b < p \leq p_a, \\ Q & \text{if } p \leq p_b. \end{cases}$$

Let F denote the distribution function of the random variable $\bar{p}(z_{-i}, s_{-i})$ conditional on $\tilde{s}_i = s_i$. Define $F^0(p) = \lim_{p' \uparrow p} F(p')$ and $\Delta F(p) = F(p) - F^0(p)$. Set $h(p) = E[\tilde{v} \mid \tilde{s}_i = s_i, \bar{p} < p]$ and $g(p) = E[\tilde{v} \mid \tilde{s}_i = s_i, \bar{p} = p]$.

Suppose first that F is continuous at p_a and p_b, so there is zero probability of bidder i being rationed. In this case the expected profits are

$$(h(p_a) - p_a)qF(p_a) + (h(p_b) - p_b)(Q - q)F(p_b). \tag{A1}$$

This is a weighted average of $(h(p_a) - p_a)QF(p_a)$ and $(h(p_b) - p_b)QF(p_b)$, which are the expected profits from bidding for the entire quantity at p_a and p_b respectively. Thus, the expected profits from the step function are no greater than the maximum of the expected profits from bidding for Q at one of the prices p_a and p_b.

Suppose now that $\Delta F(p_b) = 0$ but $\Delta F(p_a) \neq 0$. The expected profits are

$$(h(p_a) - p_a)qF^0(p_a) + (g(p_a) - p_a)\left(\frac{qQ}{Q + q}\right)\Delta F(p_a)$$
$$+ (h(p_b) - p_b)(Q - q)F(p_b).$$

The factor $qQ/(q + Q)$ is the rationed quantity from the formula (1). There exists a sequence $p^\nu \downarrow p_a$ such that F is continuous at each p^ν. If we replace p_a by p^ν, then the expected profits will change by

$$(h(p^\nu) - p^\nu)qF(p^\nu) - (h(p_a) - p_a)qF^0(p_a)$$
$$- (g(p_a) - p_a)\left(\frac{qQ}{Q + q}\right)\Delta F(p_a). \tag{A2}$$

By the right-continuity of F, this converges to

$$(g(p_a) - p_a)\left(q - \frac{qQ}{Q + q}\right)\Delta F(p_a)$$

as $\nu \rightarrow \infty$. Thus, if $g(p_a) > p_a$, the expected profits can be increased by changing p_a to a slightly greater price at which F is continuous. This avoids the rationing at price p_a. There also exists a sequence $p^\nu \uparrow p_a$ such that F is continuous at each p^ν. If we repalce p_a by p^ν, then the expected profits again change by (18). as $\nu \rightarrow \infty$, this converges to

$$-(g(p_a) - p_a)\left(\frac{qQ}{Q+q}\right)\Delta F(p_a).$$

Therefore, if $g(p_a) < p_a$, the expected profits can be increased by changing p_a to a slightly lower price at which F is continuous. Since we are seeking an optimum, it follows that there is no loss of generality in assuming $(g(p_a) - p_a)\Delta F(p_a) = 0$.

The same argument shows that we can assume $(g(p_b) - p_b)\Delta F(p_b) = 0$. In this case, the expected profits are

$$(h(p_a) - p_a)qF^0(p_a) + (h(p_b) - p_b)(Q - q)F^0(p_b).$$

This is a weighted average of the expected profits from bidding for Q at the prices p_a and p_b, so it is no greater than the maximum of the expected profits from these flat demand curves. ∎

Proof of Proposition 1. The reasoning in the proof of Theorem 1 shows that the best price on the supply curve facing any bidder is $p = p^*$. In case 1 both types of bidders want the maximum possible quantity at this price, which is $Q/2$. This price-quantity pair is realized by submitting the demand schedule (4).

In case 2 a high-type bidder again wants the maximum possible quantity, regardless of the type of the other bidder. This is realized by submitting the demand schedule (4). Consider a low-type bidder. His optimal demand is zero if the other bidder is a low type. His optimal point playing against the supply curve generated by (4) would be $p = p^*$, $q = Q/2$. On an ex ante basis he prefers zero to this point.

In cases 3 and 4, p^L lies above the conditional support of $\tilde{v}$ for a low-type bidder, so zero is clearly an optimal demand. In these cases, a high-type bidder would prefer a zero quantity whenever the other bidder is a low type and would prefer the maximum possible quantity at price p^L whenever the other bidder is a high type. On an ex ante basis the maximum quantity ($Q/2$) is optimal in case 3, and zero is optimal in case 4. The given demand curves realize these quantities in the respective cases.

To prove the last statement let p^{ab} denote the equilibrium price, and let q^{ab} denote the quantity received by a bidder when he is type a and the other bidder is type b. The self-selection conditions are

$$(v^M - p^{HL})q^{HL} + (v^H - p^{HH})q^{HH} \geq (v^M - p^{LL})q^{LL} + (v^H - p^{LH})q^{LH}, \quad \text{(A3)}$$

$$(v^L - p^{LL})q^{LL} + (v^M - p^{LH})q^{LH} \geq (v^L - p^{HL})q^{HL} + (v^M - p^{HH})q^{HH}. \quad \text{(A4)}$$

Equation (A3) states that a high-type bidder prefers the demand schedule he submits to the one submitted by a low-type bidder, and Equation (A4) states that a low-type bidder prefers his demand schedule to that submitted by a high-type bidder. Substituting $p^{LH} = p^{HL}$, $q^{HH} = q^{LL} = Q/2$, and $q^{LH} = Q - q^{HL}$ in these inequalities and rearranging gives

$$(v^H + v^M - 2p^{LH})q^{HL} \geq \frac{Q}{2}(v^H + v^M + p^{HH} - p^{LL} - 2p^{LH}), \quad \text{(A5)}$$

$$\frac{Q}{2}(v^M + v^L + p^{HH} - p^{LL} - 2p^{LH}) \geq (v^M + v^L - 2p^{LH})q^{HL}. \quad \text{(A6)}$$

Inequality (A5) implies that $q^{HL} \geq Q/2$ if $p^{HH} \geq p^{LL}$, and inequality (A6) implies that $q^{HL} \geq Q/2$ if $p^{HH} \leq p^{LL}$. Therefore, $q^{HL} \geq Q/2$.

A low-type bidder pays $p^{LL} \geq p^L > v^L$ when the other bidder is low type and receives the quantity $q^{LL} = Q/2 \geq q^{LH}$. If $q^{LL} > q^{LH}$, then a low-type bidder could increase his profit by making his demand inelastic at q^{LH} for prices less than or equal to p^{LH}. This would reduce his loss when the other bidder is a low type (by reducing the price, it might even turn the loss into a profit). Therefore, $q^{LH} = Q/2$.

Substituting $q^{HL} = Q/2$ in (A5) and (A6), we obtain $p^{LL} = p^{HH}$. Thus, both demand curves pass through the point $\hat{p} \equiv p^{LL} = p^{HH}$, $q = Q/2$. It is easy to check from the definition of an equilibrium price that $p^{LH} = \hat{p}$ also. Finally, note that the expected profit of a low-type bidder must be nonnegative, so $\hat{p} \leq (v^L + v^M)/2$. ■

Proof of Proposition 2. By Theorem 3 it suffices to show that the given price bids form an equilibrium in the first-price auction in which the only allowable quantity bid is Q.

The expected profit for a high type from a bid of Q at price $l < p \leq (l + v^H)/2$ is

$$\frac{1}{2}(v^M - p)Q + \frac{1}{2}(v^H - p)\left(\frac{p - l}{v^H - p}\right)Q = (v^M - l)\frac{Q}{2}.$$

The expected profit from a bid at $p > (l + v^H)/2$ is

$$\frac{1}{2}(v^M - p)Q + \frac{1}{2}(v^H - p)Q < (v^M - l)\frac{Q}{2}.$$

To show that the given mixed strategy is optimal, it suffices now to show that bids at or below l yield expected profit no greater than $(v^M - l)Q/2$. In both cases, bids below l are either impossible or never win. A bid at price l gives expected profits less than or equal to $(v^M - l)Q/2$ in case 1 and exactly $(v^M - l)Q/2$ in case 2.

The expected profit for a low type from a bid of Q at price $l < p \leq (l + v^H)/2$ is

$$\frac{1}{2}(v^L - p)Q + \frac{1}{2}(v^M - p)\left(\frac{p - l}{v^H - p}\right)Q < (v^L - l)\frac{Q}{2} \leq 0.$$

The expected profit from a bid at $p > (l + v^H)/2$ is

$$\tfrac{1}{2}(v^L - p)Q + \tfrac{1}{2}(v^M - p)Q < 0.$$

Bids at prices below l are either impossible or never win. A bid for any quantity at l gives zero expected profits in case 1 and hence is optimal. It gives negative expected profits in case 2, so not bidding is optimal.

The sum of the expected profits of bidders and the expected revenue of the seller must equal the expected value of the quantity sold. The expected value of the quantity sold is $(v^L/4 + v^M/2 + v^H/4)Q$ in case 1 if the low type bids for Q. The low-type bidder makes zero expected profits. The expected number of high-type bidders is one, and a high-type bidder makes expected profits of $(v^M - v^L)Q/2$, so the seller's expected revenue is

$$\left(\frac{v^L}{4} + \frac{v^M}{2} + \frac{v^H}{4} - \frac{v^M}{2} + \frac{v^L}{2}\right)Q = \left(\frac{3v^L}{4} + \frac{v^H}{4}\right)Q.$$

In case 2, the expected value of the quantity sold is $(v^M/2 + v^H/4)Q$, the low-type bidder makes zero expected profits, and the high-type bidder makes expected profits of $(v^M - p^L)Q/2$, so the seller's expected revenue is

$$\left(\frac{v^M}{2} + \frac{v^H}{4} - \frac{v^M}{2} + \frac{p^L}{2}\right)Q = \left(\frac{p^L}{2} + \frac{v^H}{4}\right)Q.$$

To see that the equilibria are unique, consider any mixed-strategy equilibrium in which both types bid for the entire quantity at some price or do not bid. Let F_L denote the distribution of the price bid of a low-type bidder and F_H the distribution of the price bid of a high-type bidder. Define $\alpha_L = \sup_p F_L(p)$ and $\alpha_H = \sup_p F_H(p)$. These can be different from one because not bidding is an option. Define

$$\underline{p}_L = \sup\{p \mid F_L(p) = 0\}, \qquad \bar{p}_L = \inf\{p \mid F_L(p) = \alpha_L\},$$
$$\underline{p}_H = \sup\{p \mid F_H(p) = 0\}, \qquad \bar{p}_H = \inf\{p \mid F_H(p) = \alpha_H\}.$$

Normalize $Q = 1$.

The expected profit for a low-type bidder from bidding a price p is

$$\tfrac{1}{2}(v^L - p)F_L^0(p) + \tfrac{1}{4}(v^L - p)\Delta F_L(p) + \tfrac{1}{2}(v^M - p)F_H^0(p) + \tfrac{1}{4}(v^M - p)\Delta F_H(p), \tag{A7}$$

where we are adopting the notation used in the proof of Theorem 3. The fractions $\frac{1}{4}$ reflect the rationing when another bidder bids the same price p. Similarly, the expected profit for a high type is

$$\tfrac{1}{2}(v^M - p)F_L^0(p) + \tfrac{1}{4}(v^M - p)\Delta F_L(p) + \tfrac{1}{2}(v^H - p)F_H^0(p) + \tfrac{1}{4}(v^H - p)\Delta F_H(p). \tag{A8}$$

Subtracting (A7) from (A8) gives

$$\tfrac{1}{2}(v^M - v^L)F_L^0(p) + \tfrac{1}{4}(v^M - v^L)\Delta F_L(p) + \tfrac{1}{2}(v^H - v^M)F_H^0(p) + \tfrac{1}{4}(v^H - v^M)\Delta F_H(p).$$

This is an increasing function of p. Hence, the argmax set of (A8) must lie above that of (A7), which implies $\underline{p}_H \geq \bar{p}_L$.

This implies that a bid $p < \bar{p}_L$ never wins against a high-type bidder, so such a bid can give nonnegative expected profits for a low-type bidder only if $p \leq v^L$. This implies $\sup\{p \mid p < \bar{p}_L\} \leq v^L$, whence it follows that $\bar{p}_L \leq v^L$. This means that a low-type bidder never bids above v^L.

The expected profit for a low-type bidder from bidding $p < v^L$ is

$$\tfrac{1}{2}(v^L - p)F_L^0(p) + \tfrac{1}{4}(v^L - p)\Delta F_L(p).$$

If $\Delta F_L(p) \neq 0$, this can be increased by bidding slightly above p at a continuity point of F_L. Hence, if $\Delta F_L(p) \neq 0$, then p is not an optimal bid for a low-type bidder, which is inconsistent with $\Delta F_L(p) \neq 0$ and F_L being an optimal strategy. So we conclude that $\Delta F_L(p) = 0$ ($\forall p < v^L$).

Suppose $\underline{p}_L < v^L$. Because F_L is continuous below v^L, the supremum in the definition of $\underline{p}_L$ is actually a maximum; that is, $F_L(\underline{p}_L) = 0$. The expected profit for a low-type bidder from any bid p in $[\underline{p}_L, \bar{p}_L)$ is $\frac{1}{2}(v^L - p)F_L(p)$. This attains its maximum in p arbitrarily close to $\underline{p}_L$ and is continuous at $\underline{p}_L$; hence it attains its maximum also at $\underline{p}_L$. It equals zero at p_L because $F_L(\underline{p}_L) = 0$. The only way the maximum can be zero is if $F_L(\underline{p}) = 0\ \forall p < \underline{v}^L$. We have already observed that a low-type bidder never bids above $\bar{v}^L$, so a low-type bidder bids v^L or does not bid. In case 1, a low-type bidder is indifferent about bidding at v^L and in case 2, bidding at v^L is not allowed, so we have verified that low-type bidders always behave as described in Proposition 2.

Because low-type bidders never bid above l, the expected profit for a high-type bidder from a bid $l < p < v^H$ is

$$\tfrac{1}{2}(v^M - p) + \tfrac{1}{2}(v^H - p)F_H^0(p) + \tfrac{1}{4}(v^H - p)\Delta F_L(p).$$

Because $p < v^H$, this could be increased by raising p slightly if $\Delta F(p) \neq 0$. Hence, it must be that F has no mass points in (l, v^H). The same argument, adjusting for possible rationing at l, shows that l cannot be a mass point either. Therefore, the expected profit is

$$\tfrac{1}{2}(v^M - p) + \tfrac{1}{2}(v^H - p)F_H(p). \tag{A9}$$

If $p_H > l$, then $F_H(\bar{p}_H + \epsilon) \approx F_H(l + \epsilon)$ for small $\epsilon > 0$, so expected profits could be increased by bidding the discretely lower price $l + \epsilon$ rather than $\bar{p}_H + \epsilon$. Since bids $\bar{p}_H + \epsilon$ are optimal for small ϵ, this implies that $\bar{p}_H = l$. For the same reason, there cannot be any gaps in the support of F_H, so F_H is a continuous function on $\Re_+$ that is strictly monotone on $[l, \bar{p}_H]$. Because F_H is an optimal strategy, this implies that (A9) is constant on $[l, \bar{p}_H]$. When $p = l$, (A9) equals $(v^M - l)/2$, so

$$\tfrac{1}{2}(v^M - l) = \tfrac{1}{2}(v^M - p) + \tfrac{1}{2}(v^H - p)F_H(p).$$

Equivalently,

$$F_H(p) = (p - l)/(v^H - p).$$

The upper bound $\bar{p}_H$ is found by solving $F_H(p) = 1$, yielding $\bar{p}_H = (l + v^H)/2$. ■

Proof of Proposition 3. Because the seller's problem is linear in the choice variables, the Kuhn–Tucker conditions are necessary and sufficient for an optimum. Let α^L, α^M, α^H, δ^L, δ^H, λ^L, and λ^H denote non-negative multipliers for constraints (6)–(12) respectively. The Kuhn–Tucker conditions are the constraints (6)–(12), the corresponding complementary slackness conditions, and the first-order conditions for maximizing the Lagrangian in the choice variables. The first-order conditions are

$$1 - \delta^L - \lambda^L + \delta^H = 0, \tag{A10}$$

$$1 + \delta^L - \delta^H - \lambda^H = 0, \tag{A11}$$

$$\delta^L v^M + \lambda^L v^L - \delta^H v^M - \alpha^L \leq 0 \quad \text{with equality if } x^{LL} > 0, \tag{A12}$$

$$\delta^L v^M + \lambda^L v^M - \delta^H v^H - \alpha^M \leq 0 \quad \text{with equality if } x^{LH} > 0, \tag{A13}$$

$$-\delta^L v^L + \delta^H v^M + \lambda^H v^M - \alpha^M \leq 0 \quad \text{with equality if } x^{HL} > 0, \tag{A14}$$

$$-\delta^L v^M + \delta^H v^H + \lambda^H v^H - \alpha^H \leq 0 \quad \text{with equality if } x^{HH} > 0. \tag{A15}$$

Equation (A10) is the first-order condition for t^{LL}, which is the same as the first-order condition for t^{LH}. Equation (A11) gives the first-order condition for t^{HL} and t^{HH}. Equations (A12)–(A15) are the first-order conditions for the quantities.

Set $x^{LH} = 0$, $x^{HL} = Q$, $x^{HH} = Q/2$, $\alpha^M = v^M$, $\alpha^H = v^H$, $\delta^L = 0$, $\delta^H = 1$, $\lambda^L = 2$, and $\lambda^H = 0$. Let the transfers and x^{LL} satisfy (13) and (14). Then all the Kuhn–Tucker conditions are satisfied except (6) and (A12) and the complementary slackness condition corresponding to (6). If $v^L \leq v^M/2$, then these are satisfied by $x^{LL} = 0$ and $\alpha^L = 0$. If $v^L \geq v^M/2$, then these are satisfied by $x^{LL} = Q/2$ and $\alpha^L = 2v^L - v^M$. ∎

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