Working Orders in Limit-Order Markets and Floor Exchanges ⋆

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Abstract

Limit-order markets and floor exchanges are analyzed, assuming an informed trader and discretionary liquidity traders use market orders and can either submit block orders or work their demands as a series of small orders. By working their demands, large market-order traders pool with small traders. We show that every equilibrium on a floor exchange must involve at least partial pooling. Moreover, there is always a fully pooling (worked-order) equilibrium on a floor exchange that is equivalent to a block-order equilibrium in a limit-order market. Our results also apply to the NYSE's proposed "hybrid market."

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The goal of this paper is to compare alternative market designs. Glosten (1994) has shown that a market with an open limit-order book is robust to competition from other markets. Our principal result is that other market types may mimic an open limit-order book and hence have the same robustness. In particular, assuming, as does Glosten (1994), perfect competition among risk-neutral liquidity providers, a uniform-price auction has an equilibrium that is equivalent in all important respects to the equilibrium of an open limit-order book. We will argue below that uniform pricing is an essential feature of a floor exchange. Thus, this establishes an equivalence between (stylized models of) limit-order markets and floor exchanges. In these stylized models, the distinction between limit-order markets and floor exchanges is that pricing in a limit-order market is discriminatory (each limit order executes at its limit price rather than the marginal price) whereas pricing on a floor exchange is uniform (all shares in a trade execute at the same price). There are other important distinctions between limit-order markets and floor exchanges that we do not model, in particular the anonymity of orders (and hence the potential for building reputations) and the extent to which orders are revealed (and hence the opportunity for front running).[1]

A market with an open limit-order book can be viewed as a screening game. In contrast, a floor exchange, in which floor traders compete to fill an order after observing its size, can be viewed as a signaling game. Signaling games tend to have many equilibria, and we will show that floor exchanges can have equilibria that are not equilibria of an open limit-order book. However, we will show that every equilibrium of a floor exchange must involve at least partial pooling of large traders with small traders.[2]

Large traders can pool with small traders in our model because we allow traders to work orders—to execute a large order as a series of small orders. This is the primary innovation of our paper. It requires a dynamic model. As far as we know, this is the first paper to compare alternative market designs with a dynamic model. Our model is a generalization of Back-Baruch (2004), which is essentially a continuous-time Glosten-Milgrom (1985) model with optimal trading by a single informed trader (or one could say that it is a continuous-time Kyle (1985) model with discrete order sizes and Poisson arrival of liquidity trades). The key condition we use from the model is that the informed trader is always willing to trade (he plays a mixed strategy, randomizing between trading and waiting). We use this first-order condition to

[1] See Benveniste, Marcus and Wilhelm (1992) for an analysis of the role of reputation on a floor exchange.

[2] More precisely, we will show this is true of each Markov equilibrium in which the informed trader's value function is monotone. We assume the monotonicity and Markovian properties in each of our results. We will not repeat this caveat continually, but see footnote 12 for some additional discussion.

compare the costs of liquidity traders (i) submitting block orders, or (ii) working orders. We assume liquidity traders have discretion to choose the cheapest way to trade. We allow traders to work orders by submitting a series of orders an instant apart, achieving execution at essentially the same time as with a block order. We show that it is never an equilibrium in a floor exchange for all traders to use block orders (there must be at least partial pooling). Moreover, the block-order equilibrium in a limit-order market, which always seems to exist, is equivalent to an equilibrium with worked orders in a floor exchange. These equilibria (specifically the block-order equilibrium in a limit-order market) are the equilibria shown by Glosten (1994) to be "inevitable."

Ask prices in limit-order markets with risk-neutral competitive liquidity providers are "upper-tail expectations," and bid prices are "lower-tail expectations." We contrast this type of market with a uniform-price auction, in which prices are expected values conditional on order size. As already mentioned, and as will be discussed further below, we believe this is a reasonable model of a floor exchange. Assuming that ask prices are expectations on a floor exchange and are upper-tail expectations in a limit-order market, it follows that prices for small orders "should" be better on a floor exchange than in a limit-order market. This is well known (see, e.g., Glosten (1994) or Seppi (1997)). The "should" here presumes a separating equilibrium—that small orders, which have less information content, can be distinguished from large orders. However, it is precisely the favorable prices for small orders on a floor exchange when traders are separated (submit block orders) that causes large traders to want to pool with small traders by working orders; i.e. to deviate from the hypothetical separating equilibrium. When orders are worked, liquidity providers on a floor exchange can of course condition on the size of an order, but they cannot condition on the size of the demand underlying the order—they cannot know whether there will be more orders from the same trader in the same direction immediately forthcoming. Thus, in a pooling equilibrium on a floor exchange, ask prices are upper-tail expectations—expectations conditional on the size of the demand being the size of the order or larger—precisely as in a limit-order market. This is the reason a pooling (worked-order) equilibrium on a floor exchange is equivalent to a block-order equilibrium in a limit-order market.

Floor exchanges can be and are organized in a variety of ways. Any floor exchange has numerous rules that are not precisely captured by our model. However, the essence of such an exchange is exposing each market order to the trading crowd, so that competition between the members of the crowd can produce the best available price for the order.[3] We assume the trading

[3] For example, NYSE rules state that market orders request execution "at the most advantageous price obtainable after the order is represented in the Trading Crowd" (cited by Hasbrouck, Sofianos and Sosebee (1993)).

crowd consists of two or more risk-neutral liquidity providers, who maximize expected profits from trade. Bertrand competition between the liquidity providers implies that the price obtained by any market order is the expectation of the asset value conditional on past information and on the size of the order. Our model most closely characterizes exchanges such as the CBOE, in which the trading crowd consists in part of market makers trading on their own accounts.[4] Of course, the risk neutrality assumption may be counterfactual, but the assumption that the trading crowd competes to fill orders after observing the size of each order seems reasonable for such exchanges.

Our results apply to (i) floor exchanges with risk-neutral competitive floor traders, (ii) combination limit-order book/floor exchanges in which limit-order traders are risk neutral and competitive and floor traders are risk neutral and competitive, and (iii) combination limit-order book/floor exchanges in which limit-order traders are risk neutral and competitive and the floor consists of a monopolist specialist. In the latter two cases, floor traders impose adverse selection on limit-order traders by being able to condition on order size. In case (ii), pricing is exactly the same as in a uniform-price auction and our results apply directly. In case (iii)—the case analyzed by Rock (1990)—it is again true that there is a worked-order equilibrium that is equivalent to a block-order equilibrium in a limit-order market. The intuition is that the informational advantage of floor traders analyzed by Rock (1990) and hence the adverse selection imposed on limit-order traders disappears when all orders are worked, because in that case neither limit-order traders nor floor traders can condition on the size of the demand underlying a market order.

Our results also apply to the New York Stock Exchange's proposed "hybrid market" (SEC File No. SR-2004-05, with seven amendments as of October 11, 2005). This proposal will increase the extent to which orders are automatically executed on the NYSE. The particular aspect of the proposal that is relevant to this study is the proposal to use uniform pricing when a market order walks up the book. Under the proposed rules, a market order will execute at the inside quote for the depth of the quote and any remainder will "sweep" the book, with the part sweeping the book all being executed at the marginal (or "clean-up") price. According to Amendment 4 (page 183),

> During a sweep, the residual shall trade with the orders on the Display Book and any broker agency interest files and/or specialist layered interest

[4] Our results apply less directly to the NYSE. The traders on the floor of the NYSE, other than the specialist, are floor brokers. While it is commonplace and sensible to regard these members of the trading crowd as providing liquidity to market orders (for example, speaking recently of proposed rule changes, John Thain, the CEO of the NYSE, stated "Floor brokers and specialists will continue to provide liquidity and make markets . . . "), they are executing orders from off-exchange traders, which should affect the prices at which they are willing to trade.

file capable of execution in accordance with Exchange rules, at a single price, such price being the best price at which such orders and files can trade with the residual to the extent possible ("clean-up price").

We will show that the hybrid market design (if there are at least four possible order sizes) has the same features as a uniform-price market: every equilibrium involves at least partial pooling, and there is a fully pooling (worked-order) equilibrium that is equivalent to a block-order equilibrium in a limit-order market.

Most of the literature on market microstructure assumes uniform pricing—i.e., it makes the same assumption that we do in our model of a floor exchange. The Kyle (1985) model and all of its variations assume uniform pricing. Most of the literature that applies the Glosten-Milgrom (1985) framework assumes single-unit demands, so there is no distinction between uniform pricing and discriminatory pricing ("discriminatory" means that each limit-order executes at its stated price, rather than at the marginal price). Two notable exceptions that discuss optimal order sizes— Easley and O'Hara (1987) and Seppi (1990)—assume uniform pricing.

We are not the first to conclude that orders should be worked in uniform-price markets. For example, in Kyle's (1985) dynamic model, the informed trader trades gradually. More closely related to our work is Chordia and Subrahmanyam (2004), who show in a two-period model with normally distributed liquidity demands and informed trading in the second period that discretionary liquidity traders should split their orders between the two periods. The contribution of our paper is to analyze the working of orders in limit-order markets and to draw the connection between worked-order equilibria in uniform-price markets and block-order equilibria in limit-order markets.

Important theoretical papers on limit-order markets, assuming discriminatory pricing, include the following:

- Rock (1990) describes the adverse selection imposed by floor traders on limit-order traders.
- Glosten (1994) demonstrates the robustness of limit-order markets against competition from other markets, assuming perfect competition in liquidity provision.
- Bernhardt and Hughson (1997) show that strategic competition between a finite number of liquidity providers using limit orders must result in positive profits for liquidity providers.
- Seppi (1997) compares a specialist market (in which the specialist faces competition in liquidity provision from floor traders) to a limit-order market, assuming perfect competition in liquidity provision.
- Ready (1999) extends Rock's (1990) analysis of the adverse selection im-

posed by a specialist on a limit-order book, assuming a single order size. His model is dynamic but limit-order traders move first and cannot kill orders that are unexecuted.[5]

- Biais, Martimort and Rochet (2000) extend Bernhardt and Hughson (1997), providing additional analysis of strategic competition between liquidity providers in limit-order markets.
- Viswanathan and Wang (2002) compare limit-order markets and uniform-price markets, assuming strategic competition between liquidity providers. In their model of a uniform-price market, a finite number of "dealers" submit demand-supply schedules before the size of the market order is known.
- Parlour and Seppi (2003) extend the analysis of Seppi (1997) to analyze competition between exchanges.
- Glosten (2003) compares limit-order markets and uniform-price markets, assuming perfect competition in liquidity provision. He endows market order traders with preferences and derives an equilibrium with optimizing market order traders as well as optimizing liquidity providers.

Of these papers, the ones most closely related to this paper are Seppi (1997), Viswanathan and Wang (2002) and Glosten (2003). Each of these compares limit-order markets to uniform-price markets. The key distinction between their analyses and the analysis in this paper is that we endogenize informed trading (and, to a certain extent, uninformed trading) in a dynamic model. Seppi (1997) and Viswanathan and Wang (2002) analyze the market structures at a point in time, taking the market order flow as given and assuming it is the same in both types of markets. Glosten (2003) points out that the market order flow should depend on the market structure, and he endogenizes it but still within a static model. Thus, the previous literature does not capture the option of working large orders as a series of small orders. As mentioned, modeling this option is the primary innovation of our paper.

We do not address the optimality of using market orders versus limit orders for traders that are motivated to trade either due to informational reasons or for liquidity reasons. Papers that address the choice between market orders and limit orders include Kumar and Seppi (1993), Chakravarty and Holden (1995), Handa and Schwartz (1996), Harris (1998), Parlour (1998), Foucault (1999), Foucault, Kadan and Kandel (2005), Goettler, Parlour and Rajan (2004, 2005) and Rosu (2005). We also do not analyze competition between exchanges, which is considered by Glosten (1994), Parlour and Seppi (1998),

[5] In contrast, we assume limit order traders continuously monitor the market, killing and resubmitting orders instantaneously. The truth obviously lies somewhere between. Interesting evidence on this point is provided by Hasbrouck and Saar (2002), who show that more than one-fourth of the limit orders on the Island ECN are killed within two seconds or less, though at least some such orders may be from liquidity demanders rather than liquidity providers.

Hendershott and Mendelson (2000) and Viswanathan and Wang (2002).

The plan of the paper is as follows. Sections I–III describe the markets assuming traders all submit block orders (Section I describes the elements of the model that are common to the two market types, Section II describes limit-order markets, and Section III describes uniform-price markets). Sections IV and V show that there are block-order equilibria in limit-order markets but no equilibria with exclusively block orders in uniform-price markets. Section VI describes equilibria in uniform-price markets in which traders work orders by submitting orders instantaneously one after the other—it shows that an equilibrium with worked orders in a uniform-price market is equivalent to a block-order equilibrium in a limit-order market, and it shows that there may be equilibria in a uniform-price market in which some orders are worked and some orders are blocks. Section VII discusses hybrid markets consisting of a floor and a limit-order book, Section VIII discusses the NYSE's hybrid-market proposal, and Section IX concludes.

I Basic Model

Our model is perhaps the simplest possible model of endogenous informed trading with multiple order sizes. It is a generalization of the model of Back and Baruch (2004).[6] We consider a continuous-time market for a risky asset and one risk-free asset with interest rate set to zero.[7] There is no minimum tick size—any real number is a feasible transaction price. There is an infinite number of risk-neutral limit-order traders or, in a uniform-price market, at least two risk-neutral traders who compete in a Bertrand fashion to fill incoming market orders. Market orders are submitted by a single informed trader and by "liquidity" traders. A public release of information takes place at a random time τ, distributed as an exponential random variable with parameter r. After the public announcement has been made, the value of the risky asset, denoted by $\tilde{v}$, will be either zero or one, and all positions are liquidated at that price.[8] All trades are anonymous. The single informed trader knows $\tilde{v}$

[6] We generalize Back and Baruch (2004) by allowing multiple order sizes and by studying limit-order markets in addition to uniform-price markets.

[7] Obviously, a one-period model would be simpler, but such a model cannot capture the ability of a trader to transact a large quantity by submitting a series of small orders. This is an important issue in the choice of order size, and it requires a dynamic model. The standard dynamic models (Kyle (1985), Glosten and Milgrom (1985)) are inadequate for our purposes, the Kyle model because it imposes a uniform-price market, and the Glosten-Milgrom model because the choice of who gets to trade at each date is made randomly rather than being determined endogenously.

[8] One can also think of the announcement date τ as a random time at which one or more traders other than the single informed trader in the model learn the in-

at date 0. If $\tilde{v} = 1$ we say the informed trader is the "high type," and if $\tilde{v} = 0$ we say the informed trader is the "low type."

We consider orders of size i for $i = 1, \ldots, n$, where n is an arbitrary but fixed integer (the units being shares or round lots, etc.). We assume buy and sell orders by liquidity traders are Poisson processes with constant arrival intensities. The arrival intensities for buy and sell orders by liquidity traders are assumed for simplicity to be the same. For orders of size i, the arrival intensity of buys and sells is denoted by β_i.

There are three parts to our definition of equilibrium:

(i) the informed trader must maximize expected profits,
(ii) limit prices and updating of conditional expectations must be consistent with Bayes' rule, and
(iii) liquidity traders must choose the method of trading—block orders or working orders—that provides the best execution.

In part (iii), the discretion we allow liquidity traders is that they can work their demands by submitting a series of orders an instant apart, thereby achieving execution at essentially the same time as with a block order. Until Section IV we will assume that liquidity traders (and hence the informed trader) submit block orders, ignoring part (iii) of the definition of equilibrium.

Our assumption about liquidity provision (part (ii) of the definition of equilibrium) implies that at a point in time pricing in each of the two market types is as follows. We focus on the buy side, the sell side being symmetric. Let a_i denote the expected value of the asset conditional on a buy order of size i and let a_{i+} denote the expected value conditional on a buy order of size i or greater. We will verify that $a_i < a_j$ for $i < j$—i.e., larger orders have more information content in equilibrium. This implies that $a_i < a_{i+}$ for each order size i. Of course, the conditional expectations depend on the trading strategies; hence, they will vary across the two market types. When necessary for clarity, we will use superscripts L and U to denote the limit-order market and uniform-price market respectively. The transaction prices in the two markets will be as follows:

Limit-Order Market For each i, there will be a limit sell order for one unit at price a^L_{i+}. The cost of a buy order of size i will be $\sum_{j=1}^{i} a^L_{j+}$.

Uniform-Price Market The cost of a buy order of size i will be ia^U_i.

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formation $\tilde{v}$ and it becomes common knowledge that this is the case. As discussed by Holden and Subrahmanyam (1992) and Back, Cao and Willard (2000), competition between identically informed risk-neutral traders will push the asset price immediately to $\tilde{v}$.

Competition between limit-order traders enforces a zero-expected-profit condition, implying that the book is as described. The inside ask must be the upper-tail expectation a^L_{1+}, because the inside limit sell order will transact against all market buy orders. Likewise, the other limit ask prices must be the corresponding upper-tail expectations.

The informed trader must play a mixed strategy in equilibrium. To see this, suppose to the contrary that the equilibrium strategy of the high-type trader calls for him to buy an amount i of the asset at a known date t (knowable from the history of trades to that date). Then the high-type trader should refrain from the purchase, causing the market to presume that the information is low and to quote a price of zero thereafter, leading hence to infinite profits. Moreover, for each order size used by liquidity traders, the high-type trader must submit a buy order of that size with positive probability and the low-type trader must submit a sell order of that size with positive probability, for each value m of m_{t-}. Otherwise, a profitable order will have no information content (or actually move the price in a desirable direction, if the low type buys or the high type sells a particular size) and hence should be used with probability one, contradicting the assumption that it is used with zero probability.

Formally, we model the trading strategy of the informed trader as a collection of $2n$ counting processes, which count the number of buys and sells of each size i through each date t. A mixed strategy means that the counting processes have arrival intensities similar to the arrival intensity of a Poisson process (though, unlike a Poisson process, the arrival intensities need not be constant).

We will look for equilibria in each market type in which the conditional expectation of the asset value is a Markov process. Thus, the expectation at any date t will be a sufficient statistic for the information in the order flow up to and including date t. We let m_t denote the conditional expectation at date t and as is customary define $m_{t-} = \lim_{s \uparrow t} m_s$. The interpretation of m_{t-} is that it is the conditional expectation just before observing whether an order is submitted at date t. This will be the state variable for the Bayesian updating of liquidity providers and the optimization of the informed trader at date t. Of course, because the asset value is either zero or one, m_t also denotes the conditional probability that the asset value is one. We assume that liquidity providers are uncertain about the asset value at the initial date ($0 < m_0 < 1$), but the precise value of m_0 is irrelevant for our results.

To describe the evolution of the conditional expectation m_t through time, we need to introduce notation for the counting processes of orders.[9] The counting processes for liquidity trades are denoted by Z, the counting processes

[9] The equilibrium order processes and the conditional expectation process m will in general depend on the market type, even though our notation does not indicate it.

for informed trades by X, and the counting process for total trades by Y ($Y = X + Z$). We use superscripts + and − to denote counting processes for buy and sell orders respectively. For example, Z^+_{it} denotes the total number of buy orders of size i by liquidity traders through time t. It jumps up by one each time a liquidity buy order of size i arrives. We are assuming that Z^+_i and Z^-_i are Poisson processes with intensities β_i. Likewise, X^+_i counts the buy orders of size i from the informed trader, and $Y^+_i = X^+_i + Z^+_i$ counts the total buy orders of size i.

We define $Z_i = Z^+_i - Z^-_i$, $X_i = X^+_i - X^-_i$, and $Y_i = Y^+_i - Y^-_i$. For example, the process Y_i jumps up by one when any buy order of size i arrives and jumps down by one when any sell order of size i arrives. Without loss of generality (given risk neutrality), we assume the informed trader has no initial position in the risky asset, so $\sum_{i=1}^n iX_{it}$ denotes the number of shares the informed trader owns at date t. The vector process $Y = (Y_1, \dots, Y_n)$ reveals the complete history of anonymous trades.

We let $a_i(m)$ $[b_i(m)]$ denote the conditional expectation of the asset given a buy [sell] order of size i at date t when $m_{t-} = m$. We denote the conditional expectation given a buy [sell] order of size i or greater as $a_{i+}(m)$ $[b_{i+}(m)]$.

The conditional expectation m_t will certainly jump up or down when an order arrives and may also evolve between transactions.[10] We can write its dynamics in either market type (dropping now the superscripts L and U) as

$$\mathrm{d}m_t = f(m_{t-})\,\mathrm{d}t + \sum_{i=1}^n [a_i(m_{t-}) - m_{t-}]\,\mathrm{d}Y^+_{it} + \sum_{i=1}^n [b_i(m_{t-}) - m_{t-}]\,\mathrm{d}Y^-_{it}\,. \tag{1}$$

This simply means that the conditional expectation jumps up to a_i when there is a buy order of size i (so $\mathrm{d}Y^+_{it} = 1$) and jumps down to b_i when there is a sell order of size i (when $\mathrm{d}Y^-_{it} = 1$) and that between transactions it evolves as $\mathrm{d}m_t = f(m_{t-})\,\mathrm{d}t$, where f is a function that is to be determined. We take 0 and 1 to be absorbing points for m, because further information cannot change beliefs that put probability one on the asset value being low or probability one on the asset value being high. *Everything in equation* (1)*—the equilibrium aggregate order process Y, the functions f, a_i and b_i, and the equilibrium conditional expectation process m—depends in general on the market type.*

[10] The conditional expectation should change between transactions because informed traders with different information will trade with different intensities. The absence of a trade indicates that the information of the trader is more likely to be consistent with a low intensity of trading than with a high intensity. Diamond and Verrecchia (1987) and Easley and O'Hara (1992) obtain a similar result, though in different models.

We will look for mixed-strategy equilibria in which the counting processes have arrival intensities that are functions of m_{t-}. By this we mean that there exist functions θ_i^+ and θ_i^- (that also depend on the market type) such that for each order size i the stochastic processes

$$X_{it}^+ - \int_0^t \theta_i^+(m_{s-}, \tilde{v})\,\mathrm{d}s \tag{2a}$$

$$X_{it}^- - \int_0^t \theta_i^-(m_{s-}, \tilde{v})\,\mathrm{d}s \tag{2b}$$

are martingales, relative to the informed trader's information. Thus, for example, when $\tilde{v} = 1$,

$$\mathrm{prob}\,(\,\mathrm{d}X_{it}^+ = 1) = E\left[\mathrm{d}X_{it}^+\right] = \theta_i^+(m_{t-}, 1)\,\mathrm{d}t\;. \tag{3}$$

II Limit Order Markets

In this section, we present some basic facts about our model of limit-order markets. We assume throughout the section that liquidity traders submit block orders—i.e., we ignore part (iii) of the definition of equilibrium.

Consider the informed trader's optimization problem. The profit earned by the informed trader on a buy order of size i at date t is

$$i\tilde{v} - \sum_{j=1}^{i} a_{j+}(m_{t-})\;. \tag{4a}$$

The summation represents walking up the book. Likewise, the profit earned on a sell order of size i is

$$\sum_{j=1}^{i} b_{j+}(m_{t-}) - i\tilde{v}\;. \tag{4b}$$

The informed trader chooses a trading strategy X to maximize the expected cumulative profits until the announcement date τ, which is

$$E\int_0^\tau \left\{\sum_{i=1}^{n}\left(i\tilde{v} - \sum_{j=1}^{i} a_{j+}(m_{t-})\right)\,\mathrm{d}X_{it}^+ + \sum_{i=1}^{n}\left(\sum_{j=1}^{i} b_{j+}(m_{t-}) - i\tilde{v}\right)\,\mathrm{d}X_{it}^-\right\}\;. \tag{5}$$

The informed trader computes this expectation knowing the value $\tilde{v}$ of the asset. Integrating with respect to the counting processes X_i^+ and X_i^- simply adds up the profit each time a buy or sell order of size i is submitted—i.e., the profit at the dates t when $\mathrm{d}X_{it}^+ = 1$ or $\mathrm{d}X_{it}^- = 1$. The informed trader chooses a trading strategy X to maximize the expectation (5) conditional on $\tilde{v}$ and subject to the dynamics (1) for m, where in (1) he takes f and the a_i and b_i to be exogenously given functions, and where in (5) he takes the a_{j+} and b_{j+} to be exogenously given functions.

In this maximization problem, we allow the informed trader to choose arbitrary counting processes. However, we search for a mixed-strategy equilibrium as defined in (2) in which $\theta_i^+(m,1) > 0$ and $\theta_i^-(m,0) > 0$ for each $m \in (0,1)$ and each order size i. This means that the high type randomizes over buying in all possible sizes and the low type randomizes over selling in all possible sizes. The first-order conditions for such a strategy to be optimal are straightforward. Because the objective function (5) is stationary, we can define the value at any date t as a function of the state variable m_{t-} and the asset value v. Let $J(m,v)$ denote the value function. An equilibrium in which the high type submits buy orders of all sizes with positive probabilities and the low type submits sell orders of all sizes with positive probabilities must satisfy the following conditions for each $m \in (0,1)$ and each order size i:

$$J(m,1) = i - \sum_{j=1}^{i} a_{j+}(m) + J(a_i(m),1)\ , \tag{6a}$$

$$J(m,1) \geq \sum_{j=1}^{i} b_{j+}(m) - i + J(b_i(m),1), \text{ with equality when } \theta_i^-(m,1) > 0\ , \tag{6b}$$

$$J(m,0) = \sum_{j=1}^{i} b_{j+}(m) + J(b_i(m),0)\ , \tag{6c}$$

$$J(m,0) \geq -\sum_{j=1}^{i} a_{j+}(m) + J(a_i(m),0), \text{ with equality when } \theta_i^+(m,0) > 0\ . \tag{6d}$$

Condition (6a) means that the optimal value for the high type can be realized by submitting a buy order of size i. The effect of submitting such an order is an instantaneous profit [11] and a continuation value of $J(a_i(m),1)$. Condition (6b) means that submitting a sell order of size i is not a strictly superior strategy for the high type and that it must be an optimal strategy when such an order is submitted with positive probability. Conditions (6c) and (6d) have analogous interpretations for the low type.

To have an equilibrium in which the informed trader buys or sells at random times, it must also be optimal for the informed trader to refrain from trading at each time. If he does so, then during an instant $\mathrm{d}t$ the announcement will occur with probability $r\,\mathrm{d}t$, and, if the announcement occurs, the value function

[11] The profit is actually realized at the announcement date τ but we have normalized the interest rate to zero.

becomes 0. An uninformed buy order of size i will arrive with probability $\beta_i\,\mathrm{d}t$, in which case m will jump to $a_i(m_{t-})$ and the value function will jump (up or down) to $J(a_i(m_{t-}),\tilde{v})$. Similarly, with probability $\beta_i\,\mathrm{d}t$ an uninformed sell order of size i will arrive and the value function will jump to $J(b_i(m_{t-}),\tilde{v})$. Finally in the absence of an announcement or an order, m will change by $f(m_{t-})\,\mathrm{d}t$ and the value function will change by

$$\frac{\partial J(m_{t-},\tilde{v})}{\partial m} f(m_{t-})\,\mathrm{d}t\,.$$

For it to be optimal for the informed trader to refrain from trading, all of these expected changes in the value function must cancel, which means that for each $m = m_{t-} \in (0,1)$ and each $v \in \{0,1\}$ we must have

$$rJ(m,v) = \frac{\partial J(m,v)}{\partial m} f(v) + \sum_{i=1}^{n} \beta_i \left[J(a_i(m),v) - J(m,v)\right] + \sum_{i=1}^{n} \beta_i \left[J(b_i(m),v) - J(m,v)\right]. \tag{7a}$$

The natural monotonicity and boundary conditions are, for all $m < m'$,

$$0 = J(0,0) < J(m,0) < J(m',0) < J(1,0) = \infty\,, \tag{7b}$$
$$\infty = J(0,1) > J(m,1) > J(m',1) > J(1,1) = 0\,. \tag{7c}$$

The monotonicity conditions mean that the informed trader earns higher expected profits when the asset is more mispriced. The boundary conditions mean that the informed trader would earn zero future profit if his type were detected and could earn infinite profit if the market were certain his type were the opposite of what it is.

Given the intensities with which the informed trader trades, it is simple to calculate the conditional expectations. For each order size i define

$$\pi_i^+(m) = m\theta_i^+(m,1) + (1-m)\theta_i^+(m,0) + \beta_i\,, \tag{8a}$$
$$\pi_i^-(m) = m\theta_i^-(m,1) + (1-m)\theta_i^-(m,0) + \beta_i\,. \tag{8b}$$

These are the arrival intensities for buy and sell orders of size i, conditional on the liquidity providers' information. A simple Bayes rule calculation (provided in the appendix) yields the following:

Proposition 1 *The expected value of the asset conditional on a buy order of size i at date t and given $m_{t-} = m$ is*

$$a_i(m) = \frac{m\theta_i^+(m,1) + m\beta_i}{\pi_i^+(m)}\,. \tag{8c}$$

Similarly, the expected value conditional on a sell order of size i is

$$b_i(m) = \frac{m\theta_i^-(m,1) + m\beta_i}{\pi_i^-(m)} \ . \tag{8d}$$

The expected value conditional on a buy order of size i or greater is

$$a_{i+}(m) = \frac{\sum_{j=i}^n \pi_j^+(m) a_j(m)}{\sum_{j=i}^n \pi_j^+(m)} \ . \tag{8e}$$

Likewise, the expected value conditional on a sell order of size i or greater is

$$b_{i+}(m) = \frac{\sum_{j=i}^n \pi_j^-(m) b_j(m)}{\sum_{j=i}^n \pi_j^-(m)} \ . \tag{8f}$$

Furthermore, the process (m_t) being a martingale relative to the liquidity providers' information implies

$$f(m) = m(1-m)\sum_{i=1}^n \left[\theta_i^+(m,0) + \theta_i^-(m,0) - \theta_i^+(m,1) - \theta_i^-(m,1)\right] \ . \tag{8g}$$

We conjecture that, perhaps under some auxiliary technical assumptions, conditions (6)–(8) are necessary for parts (i) and (ii) in the definition of equilibrium.[12] Theorem 2 of Back-Baruch (2004) generalizes easily to the present model and shows that they are sufficient conditions for (i) and (ii) to hold.[13] We will discuss part (iii) below. A consequence of these conditions is that there must be more information content in larger orders.

Proposition 2 *Assume conditions* (6)–(8) *hold (for all $1 \le i \le n$ and all $m \in (0,1)$ where this is meaningful). Then for each $j > i$ and each $m \in (0,1)$, $a_j(m) > a_i(m)$ and $b_j(m) < b_i(m)$.*

We attempted to solve conditions (6)–(8) numerically for $n = 2, 3, 4, 5$ and

[12] Conditions (6) and (8) are clearly necessary for a Markovian equilibrium (i.e., an equilibrium in which the the informed trader's intensities are functions of the conditional expectation m). We use the monotonicity conditions in (7b)–(7c), but we have been unable to prove they are necessary. The monotonicity means that the informed trader's expected profit is lower when the market becomes more certain of his type. If there are equilibria that do not have this property (which we doubt) they are certainly pathological.

[13] We have omitted here a very mild technical condition in the informed trader's optimization problem. To ensure expected profits are well-defined, a strategy is defined to be admissible in Back-Baruch (2004) if it does not incur infinite expected losses. That restriction on trading strategies should be imposed in the present model to establish the sufficiency result.

various parameter configurations, and we were successful in each case.[14] Figure 1 shows the value functions $J(m, 0)$ and $J(m, 1)$ of the low and high-type traders for the parameter values $n = 3$ and $r = \beta_1 = \beta_2 = \beta_3 = 1$. Figure 2 shows the intensities $\theta_i^+(m, 1)$ of buy orders by the high-type informed trader for the same parameter values. Figure 2 shows that $\theta_1^+ < \theta_2^+ < \theta_3^+$ for the high type, which means that large orders are used more intensively than small orders, implying that there is greater information content in large orders, as shown in Proposition 2.

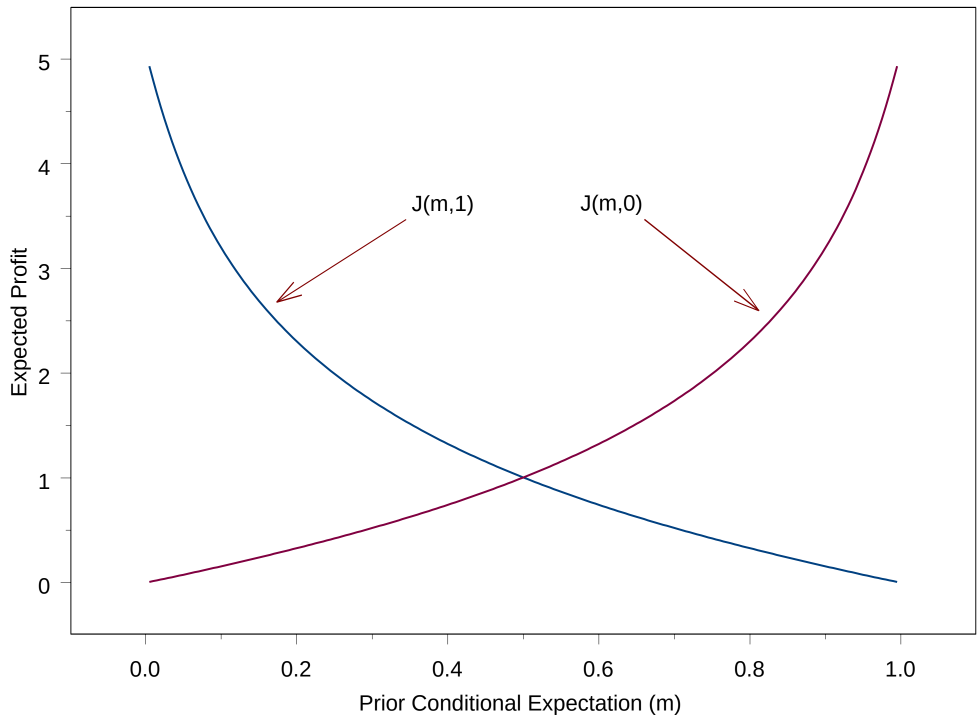

Fig. 1. Value functions. This shows the value function $J(m, \tilde{v})$ of the informed trader in a limit-order market for the parameter values $n = 3$ and $r = \beta_1 = \beta_2 = \beta_3 = 1$.

[14] The essence of the solution method is to iterate on conditions (6) and (7a). Given a guess for the value function, the equalities in condition (6) can be used to compute the ask and bid prices. Given the ask and bid prices, condition (7a) is a functional equation in the value function that can be used to update the guess. When this iteration has converged, the equilibrium order intensities of the informed trader can be computed from condition (8). We did not need to impose the monotonicity conditions in (7b)–(7c); in each case, they were automatically satisfied. Likewise, the inequality conditions in (6) were automatically satisfied with strict inequalities; i.e., there was no "bluffing" as discussed in Back-Baruch (2004). For more details, see Appendix B of Back and Baruch (2004).

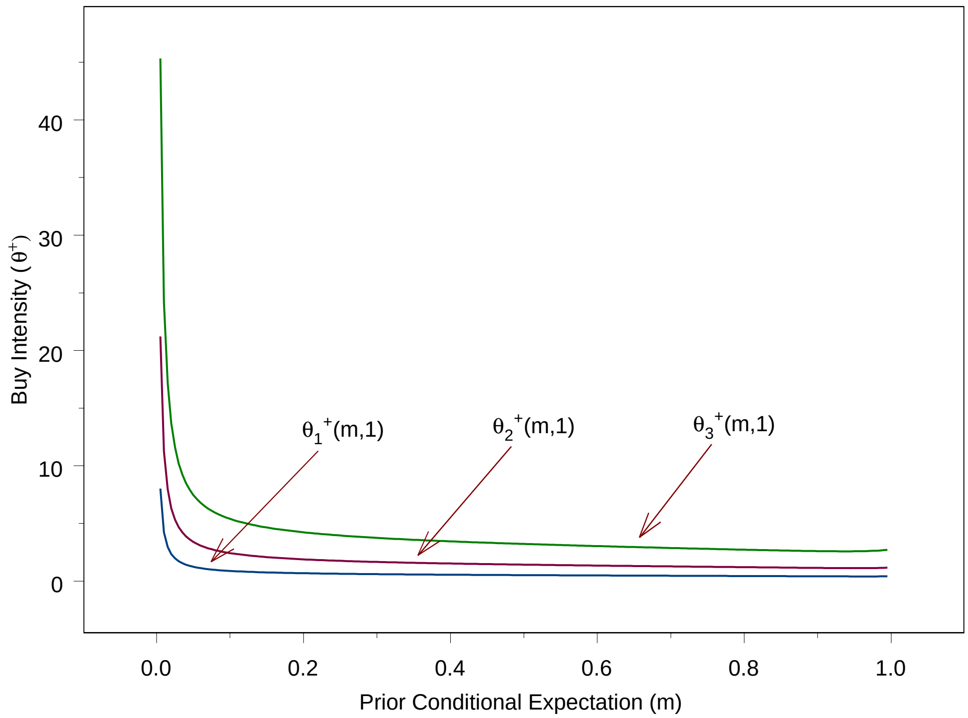

Fig. 2. Intensities of trading. This shows the intensities of different buy order sizes for the informed trader in a limit-order market when $\tilde{v} = 1$. The parameter values are $n = 3$ and $r = \beta_1 = \beta_2 = \beta_3 = 1$. The ordering of the intensities is $\theta_1^+(m, 1) < \theta_2^+(m, 1) < \theta_3^+(m, 1)$, showing that larger order sizes are used more frequently (and implying that there is more information content in larger orders). The figure also illustrates that the intensity of buying increases when m decreases.

This numerical example will be discussed further in Section V, where we show that it satisfies part (iii) of the definition of equilibrium. In fact, for each of $n = 2, 3, 4, 5$ and the various parameter configurations we have considered, the numerical solution of conditions (6)–(8) satisfies part (iii) of the definition of equilibrium. For $n = 2$ we can confirm analytically that this is true—see Section IV. Thus, it appears that there is always a block-order equilibrium in a limit-order market.

III Uniform-Price Markets

In this section, we present some basic facts about uniform-price markets. In parallel with the previous section, we assume here that liquidity traders submit block orders, ignoring part (iii) of the definition of equilibrium.

In a uniform-price market, the profit earned by the informed trader on a buy order of size i at date t is

$$i\tilde{v} - ia_i^U(m_{t-}) \ . \tag{9a}$$

Likewise, the profit earned on a sell order of size i is

$$ib_i^U(m_{t-}) - i\tilde{v} \ . \tag{9b}$$

The only difference in the equilibrium conditions in a uniform-price market compared to the equilibrium conditions (6)–(8) in a limit-order market is that the costs/revenues

$$\sum_{j=1}^{i} a_{j+}^L(m) \qquad \text{and} \qquad \sum_{j=1}^{i} b_{j+}^L(m) \tag{10a}$$

in condition (6) should be replaced by

$$ia_i^U(m) \qquad \text{and} \qquad ib_i^U(m) \ , \tag{10b}$$

respectively, in a uniform-price market. We repeat condition (6) here with these substitutions (and dropping the U superscript):

$$J(m,1) = i - ia_i(m) + J(a_i(m),1) \ , \tag{11a}$$

$$J(m,1) \geq ib_i(m) - i + J(b_i(m),1), \text{ with equality when } \theta_i^-(m,1) > 0 \ , \tag{11b}$$

$$J(m,0) = ib_i(m) + J(b_i(m),0) \ , \tag{11c}$$

$$J(m,0) \geq -ia_i(m) + J(a_i(m),0), \text{ with equality when } \theta_i^+(m,0) > 0 \ . \tag{11d}$$

The Bayes' rule calculations in Proposition 1 also apply to uniform-price markets. Furthermore, uniform-price markets share the characteristic of limit-order markets that larger orders must have more information content than small orders.

Proposition 3 *Assume conditions* (7), (8) *and* (11) *hold (for all* $1 \leq i \leq n$ *and all* $m \in (0,1)$ *where this is meaningful). Then for each* $j > i$ *and each* $m \in (0,1)$, $a_j(m) > a_i(m)$ *and* $b_j(m) < b_i(m)$.

We can solve the equilibrium conditions for the uniform-price model, ignoring part (iii) of the definition of equilibrium, in the same way that we solve the limit-order model. We present a solution for the case $n = 3$ and $r = \beta_1 = \beta_2 = \beta_3 = 1$ in Section V. However, we will show in Section V that, in contrast to the limit-order market, this numerical solution does not satisfy part (iii) of the definition of equilibrium. In fact, we will show that there is never an equilibrium in a uniform-price market in which all liquidity traders submit block orders. We begin our analysis of this issue in the next section.

IV Working Orders: Two Order Sizes

In this and the following section we focus on part (iii) of the definition of equilibrium—whether liquidity traders can reduce their execution costs by working orders. Our approach is to assume initially that liquidity traders submit block orders. Thus, the results of the previous two sections apply. We will compare the cost of a block order to the cost of working orders. We will show that it is cheaper to work orders in a uniform-price market and cheaper to submit block orders in a limit-order market. Thus, there is no block-order equilibrium in a uniform-price market, but there is a block-order equilibrium in a limit-order market.

In a limit-order market, if market buy orders are submitted with such little time between them that no new limit orders arrive and none are canceled, then the market orders will simply hit the successive limit prices, the same as if a block market order had been submitted. However, we assume that liquidity providers monitor the book continuously, so that a market-order trader need only wait an instant between orders for the book to be replenished. We will show that, nevertheless, it is optimal for market-order traders to submit block orders in a limit-order market.

The case $n = 2$ considered in this section is special because there is no possibility of partial pooling. We will present an example of a partially pooling equilibrium in a uniform-price market when $n = 3$ in Section VI. In that example, size 3 traders pool with size 1 traders and size 2 traders separate; i.e., size 3 traders work orders and size 2 traders submit blocks. In contrast, when $n = 2$ there are only two possibilities: either size 2 traders submit blocks (separate) or work orders (pool).

At any date t, the arrival of a small buy order at date t produces an updating of expectations denoted by $m_t = a_1(m_{t-})$. We abbreviate this to a_1 in the table below. If a second small buy order is submitted a very short time afterwards, then with probability arbitrarily close to one there will be no intervening order and the expected value of the asset prior to receipt of the second buy will be arbitrarily close to a_1. Receipt of the second buy will cause beliefs to change to $a_1(a_1(m_{t-}))$, which we abbreviate to $a_1(a_1)$. Using similar notation throughout, we can compare the cost of a large buy to two small buys submitted very close together in the two market types (UPM = uniform-price market and LOM = limit-order market) as follows:

Market Type	Cost of Large Order	Cost of Two Small Orders	Difference in Costs
UPM	$2a_2^U$	$a_1^U + a_1^U(a_1^U)$	$2a_2^U - a_1^U - a_1^U(a_1^U)$
LOM	$a_{1+}^L + a_2^L$	$a_{1+}^L + a_{1+}^L(a_1^L)$	$a_2^L - a_{1+}^L(a_1^L)$

In computing the costs for the limit-order market, we used the fact that $a_{2+}^L = a_2^L$, because 2 is the maximum order size. The right-hand side of the table shows the difference in the costs of a large buy versus two small buys in the two markets. Because the informed trader cares about both execution costs and the effects of trades on the market's beliefs (which determine his expected future trading profits) the critical comparison turns out to be between (i) the difference in the costs of a large buy versus two small buys and (ii) the difference in the updated expectations resulting from a large buy versus two small buys. This comparison is as follows:

Market Type	Difference in Costs Compared to Difference in Expected Values
UPM	$2a_2^U - a_1^U - a_1^U(a_1^U) \quad > \quad a_2^U - a_1^U(a_1^U)$
LOM	$a_2^L - a_{1+}^L(a_1^L) \quad < \quad a_2^L - a_1^L(a_1^L)$

To deduce the inequalities we used only the facts that $a_2^U > a_1^U$ and $a_{1+}^L > a_1^L$, which follow from Propositions 2 and 3 (large orders have more information content than small orders). The different relations ($>$ for a UPM and $<$ for a LPM) between the difference in costs and the difference in expectations are responsible for the differences in the two markets regarding the optimal behavior of discretionary liquidity traders, as we will see. A large buy is significantly more expensive than two small buys in a uniform-price market (the difference in costs is greater than the difference in expected values), so only small orders should be used in a uniform-price market; however, the opposite is true in a limit-order market.

We repeat here the first-order condition (11a) for the high-type trader in a uniform-price market for $i = 1, 2$. To reduce the notational burden, we will drop the U superscript on the a_i. Note that the value functions also depend on the market type, but our notation does not indicate that either.

$$J(m,1) = 1 - a_1 + J(a_1,1)\ , \tag{12a}$$
$$J(m,1) = 2 - 2a_2 + J(a_2,1)\ . \tag{12b}$$

The first equality holds for each m; therefore, it also holds at a_1; substituting this fact for $J(a_1,1)$ in the first line gives us

$$J(m,1) = 2 - a_1 - a_1(a_1) + J(a_1(a_1),1)\ . \tag{12c}$$

Subtracting equation (12c) from equation (12b) yields

$$J(a_2,1) - J(a_1(a_1),1) = 2a_2 - a_1 - a_1(a_1)\ . \tag{12d}$$

Equation (12d) simply says that the difference in the continuing values, from one large buy relative to two small buys, must equal the difference in the costs. Given the inequality shown in the table above we have

$$J(a_2,1) - J(a_1(a_1),1) = 2a_2 - a_1 - a_1(a_1) > a_2 - a_1(a_1)\ . \tag{12e}$$

The value function $J(\cdot,1)$ must be a decreasing function (high prices are bad for the trader with good news) so the left and right-hand sides of (12e) must have opposite signs. Therefore,

$$2a_2 - a_1 - a_1(a_1) > 0 > a_2 - a_1(a_1)\ . \tag{13}$$

The first inequality in (13) is precisely what we have claimed: two small buys are cheaper than one large buy in a uniform-price market.

The same reasoning for the limit-order market gives us the opposite result, because of the opposite inequality in the second table above. As in a uniform-price market, the difference in continuing values for the high-type trader must equal the difference in the costs (this is a consequence of the optimality condition (6a)). Thus,

$$J(a_2,1) - J(a_1(a_1),1) = a_2 - a_{1+}(a_1) < a_2 - a_1(a_1)\ . \tag{14}$$

Again as a result of $J(\cdot,1)$ being a decreasing function, the left and right-hand sides of (14) must have opposite signs. Therefore,

$$a_2 - a_{1+}(a_1) < 0 < a_2 - a_1(a_1)\ . \tag{15}$$

The first inequality in (15) shows that the cost of one large buy ($a_{1+} + a_2$) is less than the cost of two small buys ($a_{1+} + a_{1+}(a_1)$) in a limit-order market.

To further clarify the origin of the results, it may be useful to note that in both markets we have the following inequalities:

$$a_{1+}(a_1) > a_2 > \frac{a_1 + a_1(a_1)}{2} \,. \tag{16}$$

The first inequality in (16) states that a block order is cheaper in a limit-order market; the second inequality states that working orders is cheaper in a uniform-price market. These facts we have already discussed. However, the first inequality also holds in a uniform-price market—by virtue of the second inequality in (13) and the fact that $a_{1+}(a_1) > a_1(a_1)$—and the second inequality also holds in a limit-order market—by virtue of the second inequality in (15) and the fact that $a_2 > a_1$. Thus, to some extent, it is not differences in the conditional expectations in the two markets that drives the different results. Rather, it is simply the way that execution prices are determined. Prices in uniform-price markets are conditional expectations (rather than conditional tail expectations) and therefore, as noted before, "should" be better for small orders than are prices in limit-order markets. Consequently, it is cheaper to submit two small buys than one large buy in a uniform-price market, and the reverse is true in a limit-order market. To put this another way, the "unfair" prices for small orders in a limit-order market cause large orders to be incentive compatible, whereas they are not incentive compatible in a uniform-price market.

V Working Orders: The General Case

We continue to examine part (iii) in the definition of equilibrium—whether liquidity traders can obtain better execution by working orders—but now for general n. In the previous section we saw for $n = 2$ that there is a block-order equilibrium in a limit-order market[15] but no block-order equilibrium in a uniform-price market. The result for uniform-price markets extends to general n as follows: there is never an equilibrium in which all market-order traders use block orders.

For limit-order markets, we have only been able to obtain numerical results. To show that liquidity traders obtain better execution with blocks than by working orders, we need to show for each order size i and each series of order sizes $i_1, \ldots, i_k$ such that $i_1 + \cdots + i_k = i$, that the cost

$$\sum_{j=1}^{i} a_{j+}(m) \tag{17a}$$

[15] More precisely, we showed that part (iii) of the definition of equilibrium follows automatically when conditions (6)–(8) for a block-order equilibrium are satisfied.

of a buy order of size i is less than the cost

$$\sum_{j=1}^{i_1} a_{j+}(m) + \sum_{j=1}^{i_2} a_{j+}(a_{i_1}(m)) + \cdots + \sum_{j=1}^{i_k} a_{j+}(a_{i_{k-1}}(a_{i_{k-2}}(\cdots a_{i_1}(m)))) \quad \text{(17b)}$$

of submitting sequential buy orders of sizes $i_1, \ldots i_k$. As mentioned before, the numerical results indicate that the block order is always cheaper. The case of $n = 3$ can be seen from Figure 3, which uses the same parameter values as the previous figures ($r = \beta_1 = \beta_2 = \beta_3 = 1$).

Consider, for example, $m = 0.2$ (shown in Figure 3 and in Table I) and consider the cost of an order of size 3 relative to an order of size 2 and then an order of size 1. The cost of the third unit in the block order of size 3 is $a_3(0.2)$, which is 0.5668. The cost of the third unit when the order is split is $a_{1+}(a_2(0.2))$. The conditional expectation following the order of size 2 is $a_2(0.2) = 0.4191$. The inside ask quote following the order of size 2 is $a_{1+}(0.4191)$, which is $0.6710 > 0.5668$. This example generalizes—we have compared the costs (17a) and (17b) and verified numerically that the cost of the block order is smaller, for as many as five order sizes, for various values of the parameter vector $(r, \beta_1, \ldots, \beta_n)$, for each possible splitting-up of each order size, and for each value of m. Thus, we conclude that, in limit-order markets, better execution is obtained with block orders. This (apparent) fact is unsurprising: there is no benefit in breaking a large order into pieces and executing the pieces against limit prices for small orders, because those prices already anticipate execution against larger market orders. The situation in uniform-price markets is very different.

For uniform-price markets, we can establish analytically that at least some orders must be worked. The hypothesis of the following theorem must hold in any equilibrium of a uniform-price market in which all orders are block orders. The right-hand side of (18a) is the (approximate) cost of working a buy order of size i by first submitting an order of size j and then submitting an order of size $i - j$. The inequality shows that working orders is cheaper. Thus, the theorem shows, by contradiction, that there are no equilibria with exclusively block orders in a uniform-price market.

Theorem 1 *Assume conditions* (7), (8) *and* (11) *hold (for all* $1 \leq i \leq n$ *and all* $m \in (0, 1)$ *where this is meaningful). Then for each* $i > 1$*, each* $j < i$*, and each* $m \in (0, 1)$ *we have*

$$ia_i(m) > ja_j(m) + (i - j)a_{i-j}(a_j(m)) \, . \quad \text{(18a)}$$

$$ib_i(m) < jb_j(m) + (i - j)b_{i-j}(b_j(m)) \, . \quad \text{(18b)}$$

The conclusion (18) of the theorem is that it is cheaper to work any market order by splitting it into pieces. However, the theorem does not imply that all

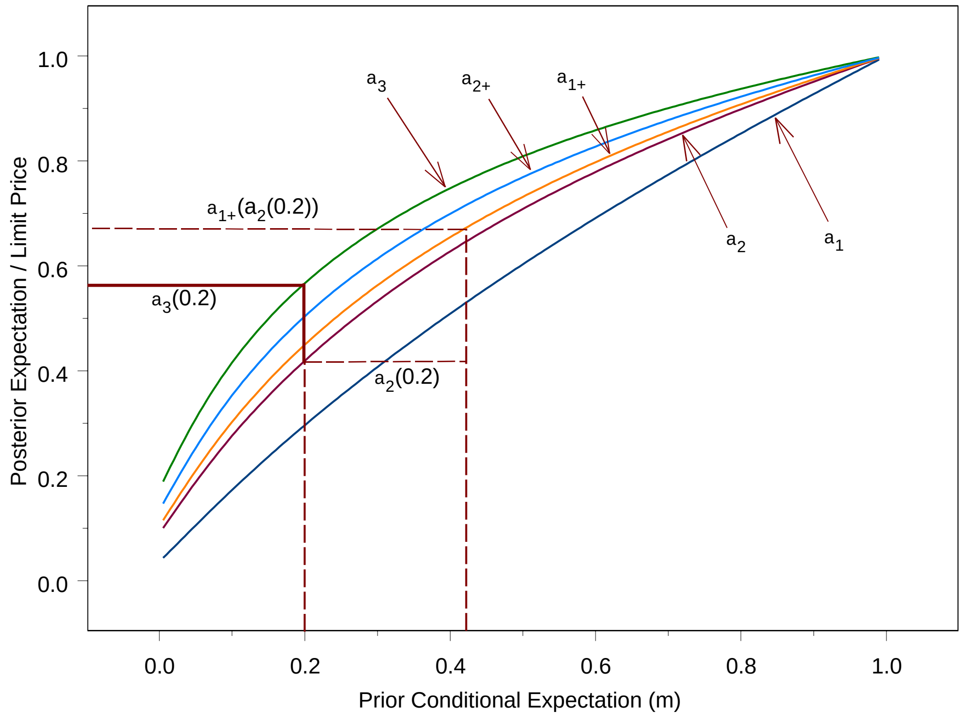

Fig. 3. Expectations and ask prices. This shows the conditional expectations a_1, a_2, a_{1+}, a_{2+} and $a_3 = a_{3+}$ in a limit-order market for the parameter values $n = 3$ and $r = \beta_1 = \beta_2 = \beta_3 = 1$. The ordering is $a_1 < a_2 < a_{1+} < a_{2+} < a_3$. The highlighted points illustrate the difference between the pricing of a block order of size 3 and the pricing of a buy order of size 2 followed by a buy of size 1, when the initial conditional expectation is $m = 0.2$. The third unit in a block buy of size 3 is priced at $a_3(0.2)$, whereas a buy of size 1 following a buy of size 2 is priced at $a_{1+}(a_2(0.2)) > a_3(0.2)$.

market orders must be worked in an equilibrium of a uniform-price market, because the hypothesis of the theorem is that the market is in a block-order equilibrium. Once we admit the possibility that some orders may be worked and some may be submitted as blocks—violating assumption (11)—the conclusion (18) need not hold. In the next section, we will give an example of an equilibrium in a uniform-price market in which some, but not all, orders are worked.

To illustrate the theorem, we can solve conditions (7), (8) and (11) and then compare the cost of block orders to the cost of working orders. Figure 4 and Table II pertain to the case $n = 3$ and $r = \beta_1 = \beta_2 = \beta_3 = 1$. Consider for example $m = 0.2$. In Figure 3 and Table 1 we showed that it is cheaper in a limit-order market (for the same parameter values) to submit a block buy of

Initial Book			Book Following a Buy of Size 2		
a_3	$=$	0.5668	a_3	$=$	0.7610
a_{2+}	$=$	0.5037	a_{2+}	$=$	0.7142
a_{1+}	$=$	0.4497	a_{1+}	$=$	0.6710
m	$=$	0.2	m	$=$	0.4191
b_{1+}	$=$	0.0923	b_{1+}	$=$	0.2150
b_{2+}	$=$	0.0777	b_{2+}	$=$	0.1837
b_3	$=$	0.0629	b_3	$=$	0.1510

Table 1
Snapshots of the limit-order book. This shows the limit-order book for the parameter values $n = 3$ and $r = \beta_1 = \beta_2 = \beta_3 = 1$. The left column is the book when $m = 0.2$. Following a buy order of size 2, the conditional expectation changes to $m = a_2(0.2) = 0.4191$. The right column is the new book at $m = 0.4191$.

size 3 than to first submit a buy of size 2 and then a buy of size 1. Figure 4 and Table 2 show that the opposite is true in a uniform-price market. The cost of a buy of size 2 when $m = 0.2$ is shown in Table 2 as $2 \times 0.4442 = 0.8884$. Following a buy of size 2, m changes to 0.4442 and the cost of a buy of size 1 is 0.5923. The total cost of this strategy is therefore $0.8884 + 0.5923 = 1.4807$. In contrast, the cost of a block buy of size 3 when $m = 0.2$ is $3 \times 0.5386 = 1.6158 > 1.4807$. In fact, submitting a buy of size 2 and then a buy of size 1 is not the cheapest way to execute a buy demand of size 3—it is even better to work the order as three successive single-unit orders. With these parameter values, the cost of three successive single-unit buy orders when $m = 0.2$ is $0.3258 + a_1(0.3258) + a_1(a_1(0.3258) = 0.3258 + 0.4732 + 0.6191 = 1.4181$. Therefore, the numerical solution shown in Figure 4 and Table 2 does not satisfy part (iii) of the definition of an equilibrium—submitting block orders is suboptimal behavior for liquidity traders.

VI A Positive Theory of Working Orders

To this point, we have only shown for uniform-price markets that there are no equilibria in which all orders are block orders; thus, some orders must be worked in equilibrium if an equilibrium exists. However, we have not presented a model of working orders. To do so, and to show that there are equilibria in

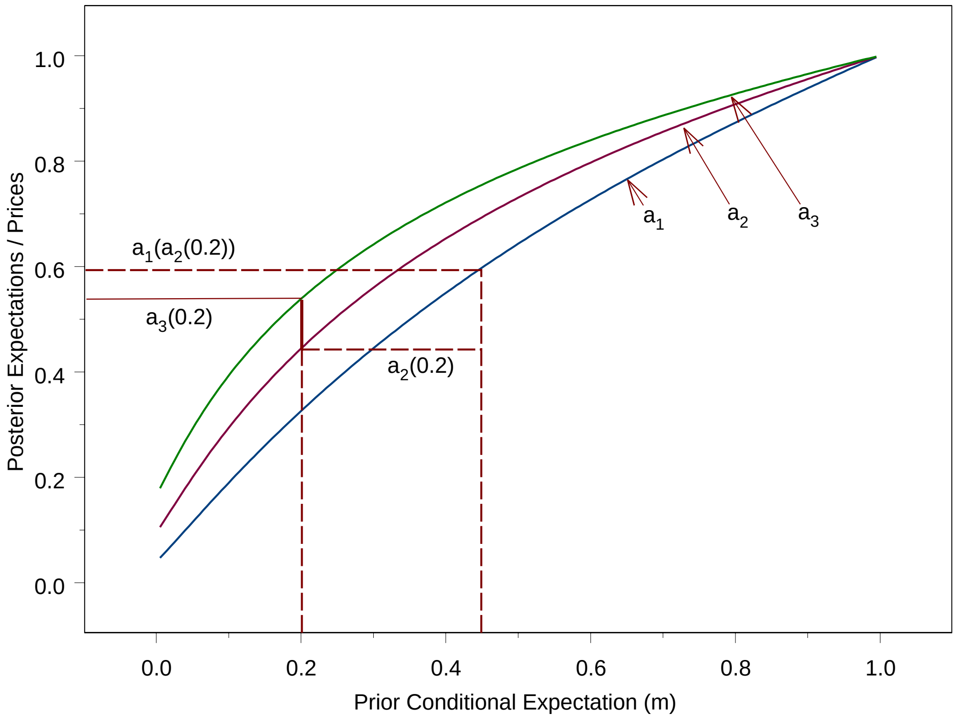

Fig. 4. Expectations and uniform prices. This shows the conditional expectations a_1, a_2, and a_3 in a uniform-price market (assuming block orders) for the parameter values $n = 3$ and $r = \beta_1 = \beta_2 = \beta_3 = 1$. The ordering is $a_1 < a_2 < a_3$. The highlighted points illustrate the difference between the pricing of a block buy order of size 3 and the pricing of a buy order of size 2 followed by buy of size 1, when the initial conditional expectation is $m = 0.2$. A buy order of size 3 is priced at $3a_3(0.2) = 1.6158$, whereas the cost of a buy of size 2 followed by a buy of size 1 is $2a_2(0.2) + a_1(a_2(0.2)) = 1.4807$.

which orders are worked, is the purpose of this section.

First, we will explain why there is always a pooling equilibrium in a uniform-price market that is equivalent to a block-order equilibrium in a limit-order market, which may be regarded as the main result of the paper. By "pooling" we mean that all traders work their orders as a series of single-unit orders. Because we have not modeled the preferences and constraints of liquidity traders, we are unable of course to determine the optimal timing of their successive single-unit orders. However, because we are working in continuous time, there is no fixed amount of time that must elapse between submission of two successive orders; thus, it is possible to assume that they obtain execution essentially at the instant they arrive at the market, even though they execute via a series of single-unit trades. This modeling choice allows liquidity traders

Initial Prices			Prices Following a Buy of Size 2		
a_3	$=$	0.5386	a_3	$=$	0.7513
a_2	$=$	0.4442	a_2	$=$	0.6889
a_1	$=$	0.3258	a_1	$=$	0.5923
m	$=$	0.2	m	$=$	0.4442
b_1	$=$	0.1270	b_1	$=$	0.3090
b_2	$=$	0.0922	b_2	$=$	0.2311
b_3	$=$	0.0722	b_3	$=$	0.1831

Table 2
Snapshots of uniform prices. This shows posterior conditional expectations (uniform prices) for the parameter values $n = 3$ and $r = \beta_1 = \beta_2 = \beta_3 = 1$, assuming block orders. The left column shows the posterior expectations when $m = 0.2$. Following a buy order of size 2, the conditional expectation changes to $m = a_2(0.2) = 0.4442$. The right column shows the posterior expectations at $m = 0.4442$.

discretion over the method of execution while retaining the assumption that they have inelastic demands at an instant in time.

In this model, liquidity providers must consider for each order the possibility that it is part of a worked order rather than representing the complete demand of the trader. Thus conditional expectations are conditioned only on the fact that the complete demand is the size of the order *or larger*. To describe this formally, we will modify our notation somewhat. We redefine $a_i^U(m)$ and $b_i^U(m)$ to denote the conditional expectations given i single-unit buy or sell orders, respectively, at any date t, when $m_{t-} = m$. The conditional expectations $a_{i+}^U(m)$ and $b_{i+}^U(m)$ have not been used in the discussion of uniform-price markets. We will define them now in a uniform-price market to mean the conditional expectation at date t given $m_{t-} = m$ and conditional on there being i or more single-unit buy or sell orders, respectively, at date t (i.e., conditional on observing i orders without knowing whether there are more orders coming at date t). The cost/revenue from submitting i single-unit buy/sell orders at date t is, with this notation,

$$\sum_{j=1}^{i} a_{j+}^U(m) \qquad \text{and} \qquad \sum_{j=1}^{i} b_{j+}^U(m)\ . \tag{19}$$

The conditional expectations a_{j+}^U and b_{j+}^U obviously correspond to limit prices in a limit-order market, because they are conditioned on the complete demand at date t being of size j or larger. The equivalence between block-order equilibria in limit-order markets and pooling (worked-order) equilibria in uniform-price markets follows almost immediately.

Theorem 2 *Let $a_i^L(m)$ and $b_i^L(m)$ be equilibrium conditional expectations in a limit-order market. Then there is an equilibrium in a uniform-price market in which $a_i^U(m) = a_i^L(m)$, $b_i^U(m) = b_i^L(m)$ for all m and in which all demands by liquidity traders and the informed trader are worked as a series of single-unit orders, with the timing of demands being the same as in the limit-order market.*

Conversely, if $a_i^U(m)$ and $b_i^U(m)$ are equilibrium conditional expectations in a uniform-price market in which all demands by liquidity traders and the informed trader are worked as a series of single-unit orders, then $a_{i+}^L(m) = a_{i+}^U(m)$ and $b_{i+}^L(m) = b_{i+}^U(m)$ are equilibrium limit-price functions in a limit-order market, and the equilibrium strategy of the informed trader is the same in the limit-order market as in the uniform-price market (with the exception that block trades in the limit-order market are executed as a series of single-unit orders in the uniform-price market).

PROOF. To establish the first part, we need to show that neither the liquidity traders nor the informed trader wish to deviate. The informed trader has more choices in this uniform-price market than in the limit-order market: he can work orders as a series of single-unit orders, in which case execution is identical to the limit-order market, or he could submit block orders. To ensure he does not wish to deviate by submitting block orders, we need to define beliefs conditional on block orders (which are out-of-equilibrium events). A particular set of beliefs that will support the equilibrium (there are many) is that the expectations upon seeing block orders of size i are $a_i^U(m)$ and $b_i^U(m)$. With these beliefs, continuation values are the same from block orders as from a series of single-unit orders, and single-unit orders have lower costs (higher revenues)—this is exactly the same as saying that walking up the limit-order book is better than executing at the marginal limit price. Therefore, the informed trader will not use block orders. The informed trader's opportunities in submitting i single-unit orders at any point in time, for various i, are exactly the same as the opportunities in submitting block orders of size i in the limit-order market. Therefore, the equilibrium behavior in the limit-order market is also equilibrium behavior (with orders worked as a series of single-unit orders) in the uniform-price market.

Liquidity traders can deviate by submitting block orders or by delaying slightly the submission of single-unit orders. With beliefs and hence prices defined as in the previous paragraph, block orders are suboptimal for liquidity traders as well as for the informed trader. The concept of delaying orders is as follows.

Consider for example submitting a series of two buy orders with the second slightly delayed. Liquidity providers will not interpret the second as being part of the same demand, so they will update expectations as $a^U_{1+}(a^U_1(m))$ rather than as $a^U_{2+}(m)$. Liquidity traders have the same opportunity in the limit-order market. If they prefer to submit a block order in the limit-order market, then it must be that $a^L_{1+}(a^L_1(m)) \geq a^L_{2+}(m)$, which, since we have defined $a^U = a^L$, implies that delaying slightly the submission of single-unit orders is not a superior strategy in the uniform-price market. The general case is the same—liquidity traders have the same opportunities in the two markets, so if they obtain the best execution with block orders in the limit-order market, they obtain best execution in the uniform-price market by submitting a series of single-unit orders that are recognized by liquidity providers as being part of the same demand.

The proof of the converse is even simpler, because, with all order sizes being used with positive probability in the limit-order market, we do not need to define out-of-equilibrium beliefs. Liquidity providers do not anticipate orders being worked as a series of single-unit orders, given the behavior prescribed for the informed trader and liquidity traders, so they update beliefs given a series of single-unit orders as $a^i_1(m)$ or $b^i_1(m)$, where i is the number of single-unit orders. The opportunities for the informed trader and the liquidity trader are the same in the limit-order market as in the uniform-price market, so the equilibrium in the uniform-price market is an equilibrium in the limit-order market.

■

Partially pooling equilibria can arise in a uniform-price market because, while mid-size traders will want to pool with small traders, they will not want to pool with large traders, so it may be equilibrium behavior for large traders to pool with small traders and for mid-size traders to separate. We will give an example of this for the parameters $n = 3$ and $r = \beta_1 = \beta_2 = \beta_3 = 1$. We have seen that there is a block-order equilibrium in a limit-order market for these parameter values (see Figures 1–3). Consequently, Theorem 2 shows that for these parameter values there is an equilibrium in a uniform-price market in which all orders are worked. There are actually two equilibria in a uniform-price market, the other equilibrium being partially pooling.

The equilibrium prices in the partially pooling equilibrium are shown in Figure 5. In this equilibrium, single-unit orders are submitted only by traders desiring to trade a single unit and traders desiring to trade three units (size 2 traders submit blocks), so it is convenient to use the symbol a_{1+} for the conditional expectation given a buy demand of size 1 or 3. Size 2 traders pay $2a_2(m)$ when submitting a block order. They could deviate to working orders, in which case the cost would be $a_{1+}(m) + a_{1+}(a_1(m))$. It turns out that the block order is cheaper, for all values of m. In fact, as Figure 5 shows, $a_{1+}(m) > a_2(m)$,[16] so it is certainly the case that $a_{1+}(m)+a_{1+}(a_1(m)) > 2a_2(m)$. The second part of a worked order by a size 3 trader is identified as coming from a size 3 trader. As before, we denote the conditional expectation given a size 3 buy demand as $a_3(m)$. Thus, the cost of submitting first a single-unit order and then an order of size 2 is $a_{1+}(m) + 2a_3(m)$. Blocks of size 3 are out-of-equilibrium events, but we can enforce the equilibrium by defining the expected value conditional on a block as being $a_3(m)$. It is obvious that working orders is cheaper than submitting a block. The size 3 trader could alternatively work his order as first a size 2 order and then a size 1 order, at cost $2a_2(m) + a_{1+}(a_2(m))$, but this is also more expensive than submitting the size 1 order first. Thus, the strategies described form an equilibrium.

Partially pooling equilibria in a uniform-price market are less robust than the fully pooling equilibrium in the sense that liquidity traders must know the exact number of possible order sizes and the other parameter values in order to know whether to submit blocks or to work orders. For example, the partially pooling equilibrium described in the previous paragraph for $n = 3$ and $r = \beta_1 = \beta_2 = \beta_3 = 1$ disappears if $n = 4$: we confirmed numerically that there is no equilibrium for these parameter values (and $\beta_4 = 1$) when $n = 4$ in which size 2 traders separate and size 3 traders pool with size 1 traders. In contrast, the fully pooling equilibrium involves all traders working orders as single-unit orders, regardless of the number of possible order sizes or other parameter values.

VII A Floor Exchange with a Limit-Order Book

Rock (1990) models a hybrid market consisting of a floor exchange and a limit-order book. In Rock's model, floor traders can provide price improvement for market orders by stepping ahead of the book by one tick (which in his model is infinitesimal) after observing the size of a market order. As is now very well understood, this "penny jumping" imposes adverse selection on limit-order traders, reducing the supply of liquidity in the limit-order book.[17]

[16] This means that the price of a single-unit buy order is more than the per-unit price of a buy order for two units; i.e., small orders have more of a price impact than mid-size orders. This is perverse but theoretically possible on a floor exchange. Reiss and Werner (1996) show that large trades get better prices than small trades on the London Stock Exchange. Partial pooling is a possible theoretical explanation, but one that seems less plausible than that given by Bernhardt, Dvoracek, Hughson and Werner (2005), since trading on the London Stock Exchange is not anonymous.

[17] Empirical evidence for this phenomenon is provided by the move to decimalization. Bessembinder (2003) shows that the frequency of price improvement on the

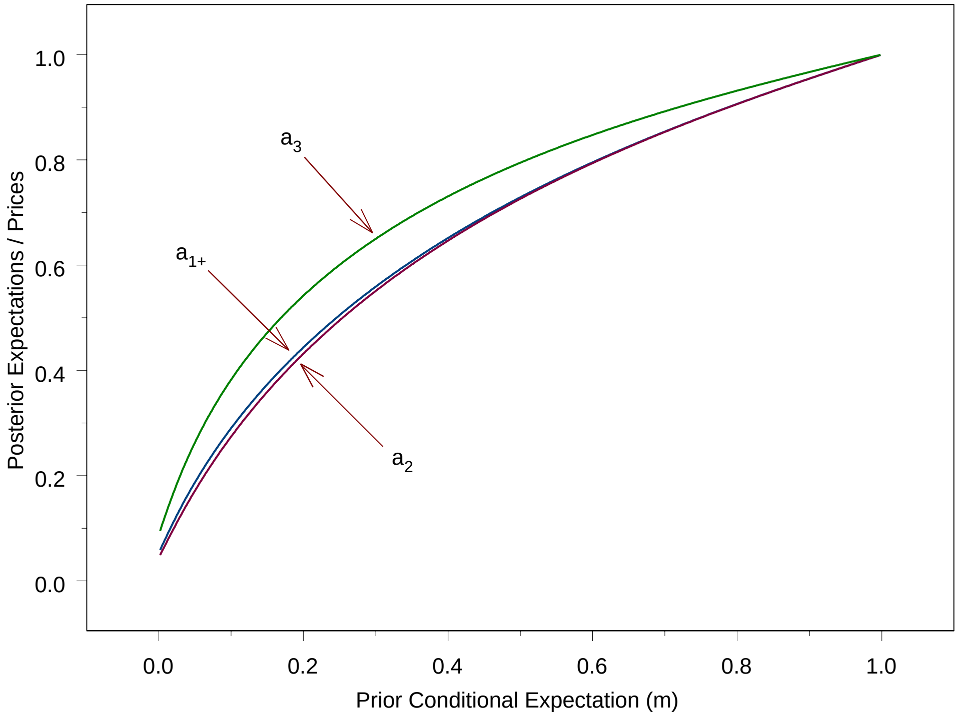

Fig. 5. Partially pooling equilibrium. This shows the partially pooling equilibrium in a uniform-price market for the parameter values $n = 3$ and $r = \beta_1 = \beta_2 = \beta_3 = 1$. A single-unit order executes at $a_{1+}(m)$ and an order for two units executes at $a_2(m)$. For these parameters, $a_2 < a_{1+}$. The size 3 trader works his buy order at cost $a_{1+} + 2a_3$.

Rock assumes the floor consists of a risk-averse monopolist specialist. If we assume the specialist is actually risk neutral, then the adverse selection imposed on the limit-order book is worsened. Consider our model but with a risk-neutral specialist. There is an equilibrium in which market-order traders all submit block orders and the execution price of each buy order is (epsilon less than) a_n and the execution price of each sell order is (epsilon more than) b_n, where, as before, a_n and b_n denote the extreme conditional expectations—the expectations conditional on the largest possible buy and sell orders respectively. In this equilibrium, it is simply impossible for limit-order traders to compete with the specialist, so the specialist makes large profits. To see this, consider submitting a limit sell order at a price $a < a_n$. Given any market

NYSE increased after decimalization, and Chakravarty, Wood and Van Ness (2004) show that the depth in the limit-order book on the NYSE declined. Goldstein and Kavajecz (2000) show that this also occurred when the NYSE moved from quoting in eighths to quoting in sixteenths.

order such that the expected asset value is $a' < a$, the specialist will price improve on the limit order to take the market order at $a - \varepsilon$ for an arbitrarily small ε. Thus, the limit sell will only execute if the expected asset value is $a' \geq a$, implying that it will lose money.

Given that every buy order executes at a_n, there is no benefit to an individual trader to work his order in this model. However, there is another equilibrium in which every market-order trader works his order and execution prices are much better. In fact, there is a worked-order equilibrium that is equivalent to a block-order equilibrium in a limit-order market. In other words, Theorem 2 also applies to a hybrid market consisting of a limit-order book and a monopolist specialist (whether the specialist is risk-averse or risk-neutral). The reason is that, when all orders are worked, the specialist has no information advantage: in a worked-order equilibrium, neither the specialist nor limit-order traders can condition on the size of the *demand* underlying a market order, and only one size of order is ever observed (the smallest size). Therefore, information is symmetric between the specialist and limit-order traders, and limit-order traders can compete effectively. Except for the inside quotes, the limit-order book is not uniquely defined in this equilibrium, but we can take it, for example, to be the same as in a block-order equilibrium of a pure limit-order market. Given this specification, market-order traders will be indifferent between submitting block orders or working orders. In equilibrium, they work orders, and they simply walk up the book when they do so.

An alternative to assuming a monopolist specialist is to assume the floor consists of competitive floor traders (perhaps in conjunction with a specialist). This is the assumption made by Seppi (1997). If the floor traders were risk neutral and there were a block-order equilibrium, then the limit-order book would be essentially empty (consisting only of offers at a_n and bids at b_n) and each market order would execute at the expected value conditional on the order size. The limit-order book would be empty for the same reason that it is empty in a block-order equilibrium when there is a monopolist specialist: floor traders will price improve whenever a limit-order is profitable, implying that every limit order, except for those at extreme prices, must lose money. Execution prices would be conditional expected values in this model because of competition between floor traders. In other words, a hybrid market consisting of a limit-order book and a competitive floor should be equivalent to a floor exchange. However, we already know (Theorem 1) that there is no block-order equilibrium in this environment. Every equilibrium must involve at least partial pooling. This affects the equilibrium limit-order book.

Theorem 2 also applies to this market: regardless of whether the floor consists of a monopolist specialist or competitive floor traders, there is a worked-order equilibrium in a hybrid market. This equilibrium is equivalent to a block-order equilibrium of a limit-order market. Except for the inside quotes, the equilibrium limit-order book is not uniquely defined. However, we can again take it to be the same as the equilibrium limit-order book in a block-order equilibrium of a pure limit-order market. In equilibrium, market-order traders work orders, and they simply walk up the book when they do so.

VIII The NYSE Hybrid Proposal

The New York Stock Exchange is frequently called a "hybrid market" because it consists of both a floor and a limit-order book. The proposal by the NYSE currently under consideration by the SEC (File No. SR-NYSE-2004-05) is to move the NYSE to being a hybrid of a floor and an *electronic open* limit-order book. The aspect of the proposal most relevant to this paper is to use uniform pricing for limit orders away from the best bid or offer. The proposal specifies (Amendment 1, p. 8): "Bids and offers on the Display Book between the displayed bid or offer and the sweep 'clean-up' price receive price improvement at the 'clean-up' price."

Clean-up pricing (i.e., marginal pricing of all but the best bids or offers) mitigates the adverse selection of limit orders executing against larger market orders (with greater information content); however, it does not mitigate the adverse selection imposed on limit orders by floor traders who can condition on order size. Assuming block orders by market-order traders, the inside offer would execute at its offer price against all market orders, so it would have to be at least as high as a_{1+}. However, the floor could step in front of this offer for single-unit buy orders, so the inside offer would actually have to be at least as high as a_{2+}. But, the floor could step in front of this for all market orders of size 2, so the inside offer would actually have to be at least as high as a_{3+}, etc. This line of reasoning, which is the same as in the previous section, shows that if block orders are used by market-order traders, then the limit order book would consist only of an offer at a_n and a bid at b_n. Market orders would execute at these extreme prices if the floor consisted of a monopolist specialist and would execute at conditional expectations if the floor included risk-neutral competitive floor traders. The upshot is that, assuming risk-neutral competitive limit-order traders, clean-up pricing should be irrelevant in a market consisting of a floor and a limit-order book.

Clean-up pricing is only one aspect of the NYSE proposal, and the overall intent of the proposal is to increase the extent of automatic execution on the NYSE. It seems useful, therefore, to consider the effects of clean-up pricing on a limit-order market, abstracting from floor brokers and the specialist. In a block-order equilibrium with risk-neutral competitive limit-order traders, the offer side of the limit-order book in this environment would consist of an offer at a_{1+} and offers for one unit each at the successive prices $a_2, \ldots, a_n$. The question we ask is whether there can be a block-order equilibrium, or whether, to the contrary, every equilibrium must involve at least partial pooling.

If $n = 2$, no limit orders in a hybrid market will receive price improvement, so the hybrid market is the same as a limit-order market. In this case, we know that there is a block-order equilibrium. The case $n = 3$ is a borderline case. We confirmed numerically that there is a block-order equilibrium in this case for the parameter values $r = \beta_1 = \beta_2 = \beta_3 = 1$. However, if $n \geq 4$, then every equilibrium must involve at least partial pooling: a proof almost identical to that of Theorem 1 shows that the inequalities (18) hold for order sizes $i \geq 4$ and to a splitting into orders of size $j \geq 2$ and $i - j \geq 2$.

It may seem obvious that orders should be worked when there is clean-up pricing. By submitting a series of small market buy orders one immediately after another, rather than a block of size i, a trader can perhaps walk up the book, executing at the successive limit prices a_{1+}, a_2, a_3, ..., a_i. In other words, a market-order trader may be able to enforce price discrimination, nullifying the clean-up pricing provisions. This is the reason usually given for why uniform pricing is infeasible in a limit-order market (see Rock (1990) or Harris (2003)). However, Theorem 1 applied to a market with clean-up pricing shows more than this: even if limit-order traders continuously monitor the market and instantaneously adjust their limit orders, so that the book may change between any two market orders (e.g., the trader submitting a series of market buy orders is not guaranteed execution at a_2 after clearing out the depth at a_{1+}), it is not an equilibrium for market-order traders to always submit block orders under the NYSE's proposed rules (when $n \geq 4$).

Regardless of whether there are block-order equilibria or equilibria with partial pooling, there is always a fully pooling (worked-order) equilibrium, which is equivalent to a block-order equilibrium in a limit-order market. In other words, Theorem 2 applies to a limit-order market with clean-up pricing (with or without a floor). This worked-order equilibrium seems particularly likely to be the equilibrium that is played in a market with clean-up pricing, given the issue discussed in the preceding paragraph.

IX Conclusion

Assuming perfect competition in liquidity provision, a single informed trader, and an infinitesimal tick size, we have established two main results: (i) every equilibrium in a uniform-price market must involve at least partial pooling of large market-order traders with small market-order traders, with pooling occurring via the working of orders by large traders, and (ii) there is a fully pooling (worked-order) equilibrium in a uniform-price market that is equivalent to a block-order equilibrium in a limit-order market. The second result depends on modeling the working of orders as submission of one order immediately after another, with no time elapsing between successive orders. However, the first result does not depend on this modeling device: even if a nonzero (but arbitrarily small) amount of time must elapse between successive orders, there is no equilibrium in a uniform-price market in which all orders are block orders. This shows that previous research on market design, which has taken order sizes to be exogenous and constant across market types, is incomplete in an important way.

The assumption that orders can be worked with no time elapsing between successive orders probably merits some further discussion. The model should be understood as the limit of a model in which orders are worked very quickly. In such a model, given that orders from other traders arrive according to intensities, two orders appearing sufficiently close together will be viewed as coming from the same trader with high probability. Thus, the second order will be priced approximately as a second order from the same trader rather than as an order from a different trader. Most microstructure models do not permit this distinction. For example, individual orders are not even observable to market makers in Kyle (1985) type models, market makers only seeing net orders at each time in such models. Furthermore, individual traders only get one opportunity to trade in the typical model based on Glosten-Milgrom (1985), so the distinction does not arise. Our model allows market makers to make this distinction, but we only solve it in the limit, where they can infer exactly when two successive orders come from the same trader. It is important to note that this is not the same as allowing liquidity providers to infer the identity of a trader—in that case, reputational considerations would come into play. Liquidity providers in our model can determine when two trades come from the same trader, but the trades are anonymous and liquidity providers do not know the identity of the trader (in particular, they do not know of course whether the trader is informed or uninformed).

It may seem somewhat counter-intuitive that traders would choose to submit successive orders very close together. In some models, rapid execution leads to poor prices. A recent and typical example is Brunnermeier and Pedersen (2005), who argue that it takes time to bring liquidity to the market, so very rapid execution leads to greater price impacts. Our model is different in that we assume perfect competition in liquidity provision. That is, we abstract from any illiquidity issues other than those arising from adverse selection. Adverse selection is enough to encourage large traders to try to "hide" the sizes of their demands—hiding (what we have called "pooling") is the essence of working orders. When doing so, a large trader has a choice about timing: whether to submit the orders close together and have them recognized as coming from a single trader, or to delay and fool the market into thinking they were submitted by different traders. The informed trader is indifferent about this choice. In equilibrium, he necessarily plays a mixed strategy, implying that it is optimal to trade at any time. In the worked-order equilibrium, liquidity traders prefer that their orders be identified as coming from the same trader. This is the perhaps counter-intuitive aspect, but it is less surprising when one considers that liquidity traders have the same choices in a limit-order market: they can submit blocks, which are orders recognized as coming from a single trader, or fool the market by working orders. If they prefer to submit blocks in a limit-order market, then they must prefer to work their orders very quickly in a uniform-price market.

A key assumption of our model, as in Glosten (1994), is perfect competition in liquidity provision. Some relevant empirical studies are Sandas (2001) and Biais, Bisiere and Spatt (2003), who show that competition by liquidity providers in limit-order markets is not perfect in the sense we assume. This is consistent with the theoretical work of Bernhardt and Hughson (1997) and Biais, Martimort and Rochet (2000), who show that perfect competition would require an infinite number of liquidity providers. Like other irrelevance propositions, our result on the equivalence of floor exchanges and limit-order markets should be viewed as pointing to the issues that matter. In addition to strategic behavior by liquidity providers, we would look to risk aversion, reputational considerations, and the potential for front running as issues that may lead to different performance of limit-order markets and floor exchanges. Our model should provide a benchmark for analyzing those issues.

A Proofs

Proof of Proposition 1 The probability of a buy order of size i arriving in an instant $\mathrm{d}t$ is

$$\pi_i^+(m)\,\mathrm{d}t = m\theta_i^+(m,1)\,\mathrm{d}t + (1-m)\theta_i^+(m,0)\,\mathrm{d}t + \beta_i\,\mathrm{d}t\ , \tag{A.1}$$

where the three terms refer to the three possible sources of an order: the high-type informed trader, the low-type informed trader, and uninformed traders. The conditional expectation is the sum of the value-weighted conditional probabilities of the order coming from each of the three possible sources, namely,

$$\frac{m\theta_i^+(m,1)}{\pi_i^+(m)} \times 1 \ + \ \frac{(1-m)\theta_i^-(m,0)}{\pi_i^+(m)} \times 0 \ + \ \frac{\beta_i}{\pi_i^+(m)} \times m\ . \tag{A.2}$$

This simplifies to equation (8c). The calculation for equation (8d) is similar, and these imply equations (8e) and (8f).

Considering only the jumps to a_i or b_i, the expected change in m in an instant $\mathrm{d}t$ given the information in the order flow and given $m_{t-} = m$ is

$$\begin{aligned}&\sum_{i=1}^{n}(a_i(m)-m)\pi_i^+(m)\,\mathrm{d}t + \sum_{i=1}^{N}(b_i(m)-m)\pi_i^-(m)\,\mathrm{d}t \\ &\qquad = m(1-m)\sum_{i=1}^{N}\Big[\theta_i^+(m,1)+\theta_i^-(m,1)-\theta_i^+(m,0)-\theta_i^-(m,0)\Big]\,\mathrm{d}t\ . \end{aligned} \tag{A.3}$$

This equation is a consequence of formulas (8c) and (8d). The process m is a conditional expectation, hence a martingale, so this expected change must be canceled by the expected change in m between transactions. This implies equation (8g).

■

Proof of Proposition 2 We will give the proof for the buy side. It suffices to show that $a_i(m) > a_{i-1}(m)$ for each i. Note that (8a), (8c) and (8e) imply $a_{i+}(m) < 1$ for each $m \in (0,1)$. From (6a) we therefore have

$$J(a_i(m),1) - J(a_{i-1}(m),1) = a_{i+}(m) - 1 < 0\ . \tag{A.4}$$

Because $J(\cdot,1)$ is a nonincreasing function, this implies $a_i(m) > a_{i-1}(m)$.

■

Proof of Proposition 3 We will give the proof for the buy side. It suffices to show that $a_i(m) > a_{i-1}(m)$ for each i. Note that (8a) and (8c) imply $a_i(m) < 1$

for each $m \in (0,1)$. From (11a) we therefore have

$$J(a_i(m),1) - J(a_{i-1}(m),1) = ia_i(m) - (i-1)a_{i-1}(m) - 1 \qquad \text{(A.5a)}$$
$$< i[a_i(m) - a_{i-1}(m)] \; . \qquad \text{(A.5b)}$$

Because $J(\cdot,1)$ is a nonincreasing function, the right-hand side of (A.5) must be positive; i.e., $a_i(m) > a_{i-1}(m)$.

■

Proof of Theorem 1 We will give the proof for the buy side. From (11a) we have

$$J(a_i(m),1) = J(m,1) - i + ia_i(m) \; , \qquad \text{(A.6a)}$$

and applying (11a) twice yields

$$J(a_{i-j}(a_j(m)),1) = J(m,1) - i + ja_j(m) + (i-j)a_{i-j}(a_j(m)) \; . \qquad \text{(A.6b)}$$

Subtracting (A.6b) from (A.6a) and using the fact that $a_j(m) < a_i(m)$ yields

$$J(a_i(m),1) - J(a_{i-j}(a_j(m)),1) = ia_i(m) - ja_j(m) - (i-j)a_{i-j}(a_j(m)) \qquad \text{(A.7a)}$$
$$> (i-j)[a_i(m) - a_{i-j}(a_i(m))] \; . \qquad \text{(A.7b)}$$

Because $J(\cdot,1)$ is a decreasing function, the left and right-hand sides of (A.7) must have opposite signs. The left-hand side being positive implies

$$ia_i(m) > ja_j(m) + (i-j)a_{i-j}(a_j(m)) \; . \qquad \text{(A.8)}$$

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